Quantum statistics and interactions

Exercise session II - C. Winkelmann - UGA/Phelma

Introduction to Quantum Fields

Let $|0\rangle$ be the vacuum state, $\{|\phi_i\rangle\}$ an orthonormal basis of single particle states, and $|\mathbf{r}\rangle$ the delta-shape wave function of a particle localized at position $\mathbf{r}$.

1 Single-particle wave function

1.1

Prove the following orthonormality and closure relations

$$\int d^3\mathbf{r}\, \phi_i^*(\mathbf{r})\phi_j(\mathbf{r}) = \delta_{ij}, \qquad \sum_i \phi_i^*(\mathbf{r})\phi_i(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}').$$

1.2

We define $\Psi^\dagger(\mathbf{r})$ as the operator creating a particle at position $\mathbf{r}$. Thus we can write $\Psi^\dagger(\mathbf{r})|0\rangle = |\mathbf{r}\rangle$. Show that $\Psi^\dagger(\mathbf{r})|0\rangle = \sum_i \phi_i^*(\mathbf{r})\, a_i^\dagger|0\rangle$.

1.3

We can thus define the field operator associated to this basis, $\Psi^\dagger(\mathbf{r}) = \sum_i \phi_i^*(\mathbf{r})\, a_i^\dagger$. Give also the adjoint operator $\Psi(\mathbf{r})$.

1.4

We consider the case of a spinless boson. Show that $\left[\Psi(\mathbf{r}), \Psi^\dagger(\mathbf{r}')\right] = \delta(\mathbf{r}-\mathbf{r}')$ and $[\Psi(\mathbf{r}), \Psi(\mathbf{r}')] = 0$.

1.5

Show that

$$N = \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}),$$

and give an interpretation of $\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})$.

1.6

We now consider the case of a spin-1/2 fermion and include the spin degree of freedom as an additional quantum number σ, writing now the operators $a_{i,\sigma}$ and $\Psi_\sigma(\mathbf{r})$. Write the relevant fermionic anticommutation relations, such as between $a_{i,\sigma}$ and $a_{j,\sigma'}^\dagger$.

1.7

We now define the field operators as

$$\Psi_\sigma^\dagger(\mathbf{r})|0\rangle = \sum_i \phi_i^*(\mathbf{r})\; a_{i,\sigma}^\dagger|0\rangle.$$

Show that $\{\Psi_\sigma(\mathbf{r}), \Psi_{\sigma'}^\dagger(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')\,\delta_{\sigma,\sigma'}$ and $\{\Psi_\sigma(\mathbf{r}), \Psi_{\sigma'}(\mathbf{r}')\} = 0$.

1.8

Define the number operator N in this case.

1.9

We consider the specific example of electrons in a cubic box of volume $V = L^3$. We use plane waves as a basis of wavefunctions

$$\phi_{\mathbf{k}}(\mathbf{r}) = \langle \mathbf{r}|\mathbf{k}\rangle = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{L^{3/2}}.$$

Give the expressions of $\Psi_\sigma(\mathbf{r})$ and $\Psi^\dagger_\sigma(\mathbf{r})$, and interpret. Eventually show that the total number operator is given by

$$N = \sum_{\mathbf{k},\sigma} a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma}.$$

2 Many-particle wave function

As above, and referring to the same single-particle basis, we can construct the many-particle wave function from the vacuum state by defining

$$|\mathbf{r}_1,\, \mathbf{r}_2,\, \ldots\rangle := \Psi^\dagger(\mathbf{r}_1)\Psi^\dagger(\mathbf{r}_2)\ldots\Psi^\dagger(\mathbf{r}_N)|0\rangle.$$

In case the particles have an internal degree of freedom σ, like the spin, this can be written

$$|\mathbf{r}_1\,\sigma_1,\, \mathbf{r}_2\,\sigma_2,\, \ldots\rangle := \Psi^\dagger_{\sigma_1}(\mathbf{r}_1)\Psi^\dagger_{\sigma_2}(\mathbf{r}_2)\ldots\Psi^\dagger_{\sigma_N}(\mathbf{r}_N)|0\rangle. \quad (1)$$

2.1

Write these bras in ket from.

2.2

Still considering the same single-particle basis as discussed previously, we construct the many-particle state

$$a^\dagger_{1,\tau_1} a^\dagger_{2,\tau_2} \ldots a^\dagger_{N,\tau_N}|0\rangle := |1\,\tau_1,\, 2\,\tau_2, \ldots, N\,\tau_N\rangle. \quad (2)$$

Show that $\langle \mathbf{r}_i\,\sigma_i|j\,\tau_j\rangle = \phi_j(\mathbf{r}_i)\,\delta_{\sigma_i,\tau_j}$.

2.3

Write the scalar product of the two vectors defined in (1) and (2), respectively.

2.4

Wick's theorem (not proven) allows stating that if there exists a permutation p transforming $(1, 2, ..., N)$ into $(i_1, i_2, ..., i_N)$, then

$$\langle 0|a_{i_N,\sigma_N}\ldots a_{i_2,\sigma_2} a_{i_1,\sigma_1} a^\dagger_{1,\tau_1} a^\dagger_{2,\tau_2}\ldots a^\dagger_{N,\tau_N}|0\rangle = \mathrm{sign}(p) \times \delta_{\sigma_1,\tau_{p(1)}}\ldots\delta_{\sigma_N,\tau_{p(N)}}.$$

What would the value of the above matrix element if there was no such permutation?

2.5

Write the scalar product from question 2.2 in the form of a Slater determinant.