Quantum statistics and interactions

Exercise session III - C. Winkelmann - UGA/Phelma

Rutherford scattering

We consider the elastic scattering of particle with charge q_1, initial energy $E = \frac{\hbar^2 k^2}{2\mu}$, and initial wave vector $\mathbf{k} \parallel \mathbf{u}_z$, from a static charge q_2.

The Coulomb potential writes $V(\mathbf{r}) = \frac{V_0}{r}$, with $V_0 = q_2/(4\pi\epsilon_0)$. Consequently, $q_1 V(\mathbf{r})$ is the potential energy. Classically, the Coulomb scattering cross section between the two charges is given by the Rutherford formula

$$\sigma_{\mathrm{Coulomb}}(\theta) = \left(\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{4E \sin^2(\theta/2)} \right)^2 ,$$

with θ the direction of the scattering (that is, the detector position) with respect to the incident beam. We will verify this result to remain valid from quantum considerations.

I Coulomb and Yukawa potentials

a.) Show that the direct calculation of the Fourier transform of V

$$\tilde{V}(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \int V(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{r}$$

doesn't converge.
b.) We can introduce a convergence factor in the potential, which now writes

$$V(\mathbf{r}) = \frac{V_0}{r} \, e^{-\alpha r}.$$

This type of potential is called *Yukawa potential*, and it has a finite range $r_0 = \alpha^{-1} > 0$. Calculate $\tilde{V}(\mathbf{q})$ with a finite α, and then take $\alpha \to 0$.

Hint: you might want to use this relation

$$\int_0^{+\infty} \sin(qr) \exp(-\alpha r) \, dr = \frac{q}{q^2 + \alpha^2}.$$

II Green's function and Born approximation

Our aim is to show that the scattering of a plane wave with wave vector $\mathbf{k} \parallel \mathbf{u}_z$ from a potential V will lead to a scattered wave function given by

$$\varphi(\mathbf{r}) = e^{ikz} + f(\theta, \varphi) \frac{e^{ikr}}{r},$$

with θ and φ the usual spherical coordinates angles.

a.) Show that we are led to seeking solutions of the stationary scattering equation

$$\left(\Delta + k^2 - U(\mathbf{r})\right)\varphi(\mathbf{r}) = 0,$$

with $U = \frac{2\mu}{\hbar^2}V$ and V a potential that goes sufficiently quickly to zero at infinity.

b.) In spherical coordinates, the Laplacian of a function depending solely on r can be written for $r \neq 0$

$$\Delta = \frac{1}{r}\frac{d^2}{dr^2}r.$$

Justify that $\forall r \neq 0, \quad \Delta(1/r) = 0.$

c.) Write the Maxwell equation for the electrostatic field created by a point charge at position $\mathbf{r} = \mathbf{0}$. Conclude that $\forall \mathbf{r}$

$$\Delta\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r}).$$

d.) We define the function

$$G(k, \mathbf{r}) = -\frac{1}{4\pi}\frac{e^{ikr}}{r}.$$

In order to regularise G at the origin, we decompose it into two functions having individually a well-defined Laplacian at 0:

$$\frac{e^{ikr}}{r} = \frac{e^{ikr} - 1}{r} + \frac{1}{r}.$$

Show that

$$(\Delta + k^2)G = \delta(\mathbf{r}).$$

This defines G as the *Green's function* of the propagation equation.

e.) Consider a wave function $\varphi(\mathbf{r})$ satisfying

$$\varphi(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}')U(\mathbf{r}')\varphi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'.$$

Show that φ is a solution of the stationary scattering equation.

f.) Show that the same is true for $\varphi(\mathbf{r})$ defined by

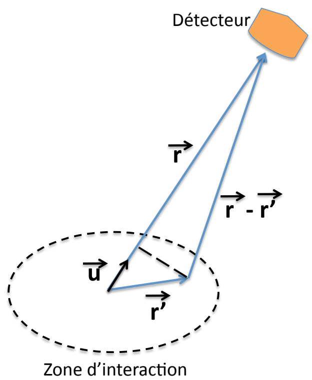
$$\varphi(\mathbf{r}) = \varphi_0(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}')U(\mathbf{r}')\varphi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}',$$

with $\varphi_0(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ or any other function satisfying $(\Delta + k^2)\varphi_0(\mathbf{r}) = 0$.

This self-consistent integral equation is equivalent to the Schrödinger equation of the stationary scattering states. When the interaction U is weak, it is usually possible to consider the integral term (scattering term) as a perturbation with respect to the non-scattered incident wave $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikz}$. The Born approximation then consists in replacing $\varphi(\mathbf{r})$ by $\varphi_0(\mathbf{r})$ inside the integral.

g.) Let us assume the detector to be at a position $\mathbf{r}$, which is very far from the interaction zone, such that $\mid \mathbf{r} \mid \gg \mid \mathbf{r}' \mid$. Show that $\mid \mathbf{r} - \mathbf{r}' \mid \approx r - \mathbf{u}\cdot\mathbf{r}'$ (see drawing).

h.) Write the approximate expression of $G(\mathbf{r} - \mathbf{r}')$ under these assumptions and derive the scattering amplitude $f(\theta, \varphi)$ within the Born approximation.



III Rutherford formula

a.) Calculate the differential scattering cross section $\sigma(\theta, \varphi)$ of a plane wave from a Yukawa potential $(\alpha \neq 0)$ in the Born approximation.

b.) Give the total scattering cross section in the form of an integral, which you do not need to calculate.

c.) To recover the Coulomb potential case, take $\alpha \to 0$ in the expression of $\sigma(\theta, \varphi)$. Compare with the classical Rutherford formula.