# Quantum statistics and interactions

Exercise session IV - C. Winkelmann - UGA/Phelma

## Jaynes-Cummings Hamiltonian and vacuum Rabi oscillations

This session deals with coupled eigenstates of light and matter. Part I introduces equivalent formulations of quantum dynamics. Part II establishes the general Hamiltonian of light-matter interaction, in the so-called Jaynes-Cummings model. In part III we will seek the eigenstates of this coupled system, leading to the notion of *dressed atom*.

Rabi oscillations between these eigenstates were experimentally first observed by the team of S. Haroche et J.-M. Raimond (M. Brune et al., Phys. Rev. Lett. 76, 1800 (1996)).

**Definitions and conventions:**

- $I_0$ is the identity operator; $e$ is the elementary charge; $[A, B] = AB - BA$ is the commutator of $A$ and $B$.
- We define the following matrices
  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ et $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
- The exponential of an operator $A$ is defined as

  $$\exp(A) = e^A = \sum_{n=0}^{+\infty} \frac{A^n}{n!}.$$

  The exponential of an operator has most of the properties of the scalar exponential function. In particular, if $A$ and $B$ commute,

  $$e^A \, e^B = e^{A+B}.$$

  Further, if $A$ is time dependent, then

  $$\frac{d \exp(A(t))}{dt} = \frac{dA(t)}{dt} \exp(A(t)).$$

  Finally, if $[A, B] = 0$ then $\left[e^A, B\right] = 0$

- As the origin of energies is arbitrary, two Hamiltonians differing only by $\lambda I_0$ (with $\lambda$ a constant) can be considered as identical.

# 1 Schrödinger, Heisenberg and Dirac pictures

Let $|\varphi(t)\rangle$ be the normalised quantum state of a particle at time $t$, living in a Hilbert space $\mathcal{H}$ and governed by a time-independent total Hamiltonian $H$. We assume that there is an operator $U(t, t_0)$ acting on $\mathcal{H} \to \mathcal{H}$ such that for all $(t, t_0)$

$$|\varphi(t)\rangle = U(t, t_0) \, |\varphi(t_0)\rangle. \tag{1}$$

We call $U$ the evolution operator: its knowledge allows determining simply $|\varphi(t)\rangle$ at any time from its value at the initial time $t_0$.

1.1 Show that under the above assumptions

$$U(t,t_0) = \exp\left(-\frac{i(t-t_0)H}{\hbar}\right)$$

fulfills the above criterion. Explain why things are more complicated when $H$ is time dependent.

1.2 Let $\langle A\rangle(t)$ be the quantum average at time $t$ of an observable $A$ (which doesn't have an explicit time dependence itself), averaged over state $|\varphi(t)\rangle$. Show that

$$\langle A\rangle(t) = \langle\varphi(t_0)|\, A_H(t)\, |\varphi(t_0)\rangle \tag{2}$$

where

$$A_H(t) := U(t,t_0)^\dagger A\, U(t,t_0).$$

Rather than postulating that operators are constant and state vectors depend on time via the Schrödinger equation, one can thus just as well postulate that the quantum state is time independent and that the observables depend on time, like $A_H$. This point of view is called the **Heisenberg picture**, as opposed to the more common **Schrödinger picture**.

There exists a third point of view, the so-called **Dirac or interaction picture**, which is very useful when $H$ contains a constant well-understood principal term $H_0$ plus a smaller interaction term $W(t)$, which may be time dependent, so that $H = H_0 + W(t)$. We define now for any state vector $|\varphi(t)\rangle$ and any operator $A(t)$ (which could be time dependent)

$$|\tilde\varphi(t)\rangle = e^{itH_0/\hbar}|\varphi(t)\rangle,$$
$$A_D(t) = \exp\left(\frac{itH_0}{\hbar}\right) A(t)\, \exp\left(-\frac{itH_0}{\hbar}\right)$$

where we have set $t_0 = 0$ for simplicity. Note that $A_D(0) = A(0)$.

1.3 Prove the so-called Schwinger-Tomonaga equation

$$i\hbar\frac{d|\tilde\varphi\rangle}{dt} = W_D|\tilde\varphi\rangle,$$

which is the equivalent of the Schrödinger equation in the interaction picture.

1.4 If $A$ is time independent (in the Schrödinger picture), show that

$$i\hbar\frac{dA_D(t)}{dt} = \left[A_D(t), H_0\right].$$

1.5 Let $H_0 = \alpha\hbar\,\sigma_z$, with $\alpha$ a real constant. Show that

$$\sigma_{+,D}(t) = e^{2i\alpha t}\sigma_+.$$

Similarly, one finds $\sigma_{-,D}(t) = e^{-2i\alpha t}\sigma_-$.

1.6 Let $a$ and $a^\dagger$ be the annihilation/creation operators appearing in the Hamiltonian $H_0 = \beta\hbar\, a^\dagger a$ of a harmonic oscillator, with $\beta$ a real constant. Show that

$$a^\dagger_D(t) = e^{i\beta t}a^\dagger.$$

Similarly, one finds $a_D(t) = e^{-i\beta t}a$.

## 2 Jaynes-Cummings Hamiltonian

We will now investigate the interaction of an atom with a single mode of the electromagnetic field, with angular frequency $\omega$, inside a cavity. The atome has only two energy levels, noted $|g\rangle$ and $|e\rangle$ (for ground and excited), which are resonant with the cavity, such that $\omega_0 := (E_e - E_g)/\hbar \approx \omega$, whereas all other electronic transitions are off resonant. We can thus neglect the existence of all atomic levels other than $|g\rangle$ and $|e\rangle$. We assume that the orbital wave functions of the eigenstates of the atomic Hamiltonian $H_{at}$ are either inversion symmetric or asymmetric : $\psi_{g,e}(-\vec{r}) = \pm\psi_{g,e}(\vec{r})$.

2.1 In the subspace generated by $|g\rangle$ and $|e\rangle$, write the matrices $\sigma_z$, $\sigma_+$ and $\sigma_-$ as a function of $|g\rangle\langle g|$, $|g\rangle\langle e|$, $|e\rangle\langle g|$ and $|e\rangle\langle e|$.

2.2 Show that the atomic Hamiltonian can be reduced to the form $-\frac{\hbar\omega_0}{2}\sigma_z$ in the basis of $(|g\rangle, |e\rangle)$.

We assume that the electric field operator in the cavity can be written

$$\vec{\mathcal{E}} = \vec{u}_z\sqrt{\frac{\hbar\omega}{\epsilon_0\nu}}\left(a + a^\dagger\right)\,\sin ky,$$

where $a^\dagger$ and $a$ are the creation/annihilation operators of a photon with energy $\hbar\omega$, $\epsilon_0$ the vacuum dielectric constant, $\nu$ the volume of the cavity and $k$ the modulus of the wave vector. The position $y$ inside the cavity will be regarded as a classical variable. The Hamiltonian of the single mode of the electromagnetic field is then

$$H_{em} = \hbar\omega\left(a^\dagger a + \frac{I_0}{2}\right)$$

and we write $|n\rangle$ the eigenstate of $n$ photons in the cavity. The term $\hbar\omega I_0/2$ drops out in the remainder.

The dipolar coupling between the atom and the electromagnetic field writes

$$W = -\vec{D}\cdot\vec{\mathcal{E}},$$

with $\vec{D}$ the quantum operator associated to the dipole moment of the atom, defined as

$$\vec{D} = -e\vec{R},$$

with $\vec{R}$ the position operator.

2.3 Justify that

$$\langle g|\vec{D}|g\rangle = \langle e|\vec{D}|e\rangle = \vec{0}.$$

2.4 We thus define $d := \langle e|\left(\vec{D}\cdot\vec{u}_z\right)|g\rangle$ (which can be taken real without any loss of generality). Show that $\vec{D}\cdot\vec{u}_z = d\,(\sigma_+ + \sigma_-)$ and thus

$$W = \frac{\hbar\,\kappa(y)}{2}\,(\sigma_+ + \sigma_-)(a + a^\dagger), \tag{3}$$

with $\kappa(y)$ a real function.

2.5 We are thus led to writing $H = H_0 + W$ with

$$H_0 = H_{at} + H_{em} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^\dagger a.$$

Express $W$ in the interaction picture, and factorise scalar time-dependent terms.

2.6 The Jaynes-Cummings model consists in neglecting the operators which have a rapid time dependence in the interaction picture. Remember that the frequency mismatch $\delta = \omega_0 - \omega$ obeys $|\delta| \ll \omega_0, \omega$. Show that the Jaynes-Cummings Hamiltonian $H_{JC}$, which describes a two-level atom coupled to a single mode of the electromagnetic field, is given by

$$H_{JC} = -\frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + \frac{\hbar\,\kappa(y)}{2}\,(\sigma_+ a^\dagger + \sigma_- a).$$

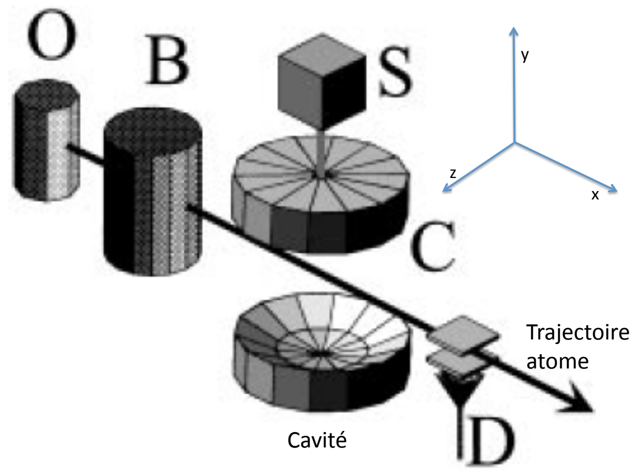


Figure 1: Experimental setup: The oven O emits rubidium atoms, which are prepared in state $|e\rangle$ in B (box). They fly through the cavity C (2 superconducting mirrors). An ultra-stable field source S feeds 0 to $n$ photons into C. The state of the atom after it has crossed C is measured in the detector D.

# 3 Dressed atom

In the experiment sketched in Fig. 1, the atom is injected to the cavity along a trajectory $\perp \vec{u}_y$, with $\sin ky = 1$ (thus $\kappa(y) = \kappa$ is a constant).

In the absence of dipolar coupling, the eigenstates of the atom and cavity states are given for example by $|e, n\rangle$, with $|e\rangle$ the state of the atom and $|n\rangle$ the state of the cavity.

3.1 Write the action of $H_{JC}$ on vectors $|e, n\rangle$ and $|g, n+1\rangle$.

3.2 Show that $H_{JC}$ can be decomposed as $H_{JC} = \sum_n H_n$, where each $H_n$ acts on a subspace generated by $(|e, n\rangle, |g, n+1\rangle)$.

3.3 Show that in the basis $(|e, n\rangle, |g, n+1\rangle)$,

$$H_n = E_n^0 I_0 + \frac{\hbar}{2} \begin{pmatrix} \delta & \kappa\sqrt{n+1} \\ \kappa\sqrt{n+1} & -\delta \end{pmatrix},$$

with $E_n^0$ a real number depending only on $n$ and $\omega$.

3.4 Find the eigenvalues of $H_n$, introducing the Rabi (angular) frequency $\Omega_n = \sqrt{\delta^2 + \kappa^2(n+1)}$.

3.5 We define the interaction angle $0 < \theta_n < \pi/2$ by

$$\sin(\theta_n) = \frac{\kappa\sqrt{n+1}}{\sqrt{(\Omega_n - \delta)^2 + \kappa^2(n+1)}}.$$

Show that the two vectors defined by

$$|\chi_n^-\rangle = \cos\theta_n |e, n\rangle - \sin\theta_n |g, n+1\rangle,$$
$$|\chi_n^+\rangle = \sin\theta_n |e, n\rangle + \cos\theta_n |g, n+1\rangle$$

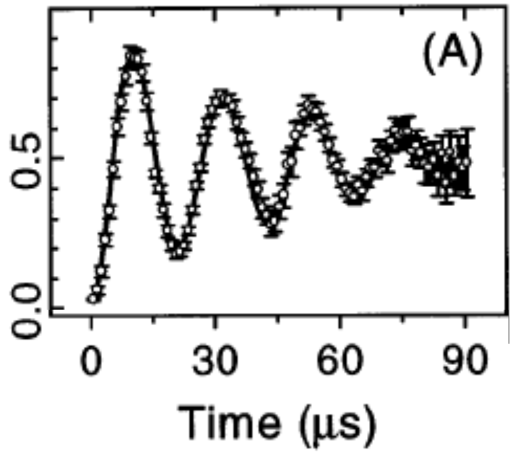


Figure 2: Experimental probability of finding the atom in state $|g\rangle$, as a function of the interaction time inside the cavity ($\delta = 0$, $n = 0$).

are an orthonormal diagonalisation basis of $H_n$.

3.6 Resonant case. We assume in this question that $\delta = 0$ ($\omega_0 = \omega$). Find the eigenvalues and -vectors of $H_n$.

3.7 Non-resonant case. Supposing now $|\delta| \gg \kappa\sqrt{n+1}$, approximate the eigenvalues and -vectors of $H_n$, depending on if the resonance mismatch is blue shifted ($\delta < 0$) or red shifted ($\delta > 0$).

3.8 Avoided crossing. Plot the eigenvalues as a function of $\delta$ and give the eigenvectors at appropriate places. Comment.

3.9 **Rabi vacuum oscillations**. Assuming resonant conditions ($\delta = 0$) and that the atom enters the cavity at $t = 0$, in its excited state $|e\rangle$, while the cavity is in its photon ground state ($n = 0$). Show that the probability to find the atom in the excited state at time $t > 0$ is

$$\mathcal{P} = \cos^2 \frac{\Omega_0 t}{2}.$$

3.10 Comment on the measurements of Brune and coworkers (Fig. 2).

3.11 Same question as in 3.9, but for $|\delta| \gg \kappa$: justify that the Rabi oscillations are strongly suppressed and that $\mathcal{P}$ remains close to 1.