Magnons

I

$$H = \underbrace{-g\mu_B B \sum_q S_q^z}_{\text{para}} \underbrace{- J \sum_q \vec{S}_q \cdot \vec{S}_{q+1}}_{\text{ferro}}$$

1.1 $\vec{S}_q \cdot \vec{S}_{q+1} = S_q^x S_{q+1}^x + S_q^y S_{q+1}^y + S_q^z S_{q+1}^z$

$$S_q^+ S_{q+1}^- + S_q^- S_{q+1}^+ = (S_q^x + i S_q^y)(S_{q+1}^x - i S_{q+1}^y) + (S_q^x - i S_q^y)(S_{q+1}^x + i S_{q+1}^y)$$

$$= S_q^x S_{q+1}^x + S_q^y S_{q+1}^y + i S_q^y S_{q+1}^x - i S_q^x S_{q+1}^y + S_q^x S_{q+1}^x + S_q^y S_{q+1}^y - i S_q^y S_{q+1}^x + i S_q^x S_{q+1}^y$$

$$= 2 (S_q^x S_{q+1}^x + S_q^y S_{q+1}^y)$$

1.2 $|0\rangle = |\uparrow\uparrow\uparrow \ldots\ldots \uparrow\rangle$

$$H|0\rangle = H_{para}|0\rangle + H_{ferro}|0\rangle$$

$$H_{para}|0\rangle = -g\mu_B B \sum_q S_q^z |0\rangle$$

$$= -g\mu_B B \sum_q \frac{1}{2}|0\rangle$$

$$= -\frac{g\mu_B B N}{2}|0\rangle$$

$$H_{ferro}|0\rangle = -J\sum\left(S_q^zS_{q+1}^z + \frac{1}{2}\left(S_q^+S_{q+1}^- + S_q^-S_{q+1}^+\right)\right)|0\rangle$$

$$= -J\sum\left(\frac{1}{4}|0\rangle + 0\right)$$

$$= -\frac{JN}{4}|0\rangle$$

$$E_0 = -\frac{g\mu_B BN}{2} - \frac{JN}{4}$$

2.1 $$|q\rangle = |\uparrow\uparrow\uparrow\ldots\uparrow\underset{\substack{\uparrow\\q}}{\downarrow}\uparrow\ldots\uparrow\rangle$$

$$H|q\rangle = -g\mu_B B\left((N-1)\cdot\left(+\frac{1}{2}\right) + 1\left(-\frac{1}{2}\right)\right)|q\rangle$$

$$-J\left(\left((N-2)\left(+\frac{1}{4}\right) + 2\left(-\frac{1}{4}\right)\right)|q\rangle\right.$$

$$\left.+\frac{1}{2}\sum_{q'}\left(S_{q'}^+S_{q'+1}^-|q\rangle + S_{q'}^-S_{q'+1}^+|q\rangle\right)\right)$$

$$= -g\mu_B B\left(\frac{N-2}{2}\right)|q\rangle - J\left(\frac{N-4}{4}\right)|q\rangle$$

$$-\frac{J}{2}\left(|q+1\rangle + |q-1\rangle\right)$$

$$= \left(E_0 + g\mu_B B + J\right)|q\rangle - \frac{J}{2}\left(|q+1\rangle + |q-1\rangle\right)$$

2.2

$$|u_k\rangle = \frac{1}{\sqrt{N}} \sum_q e^{ikqa} |q\rangle$$

$$\langle u_k | u_k\rangle = \frac{1}{N} \sum_{qq'} e^{-ikqa} e^{ikq'a} \underbrace{\langle q|q'\rangle}_{\delta_{qq'}}$$

$$= \frac{1}{N} \sum_q 1 = 1$$

2.3

$$e^{ik(N+1)a} = e^{ika}$$

$$\Rightarrow kNa = 2\pi n \qquad n \in \mathbb{Z}$$

$$k = \frac{2\pi}{Na} n$$

$n+N$:

$$|u_{k'}\rangle = \frac{1}{\sqrt{N}} \sum_q e^{ik'qa} |q\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_q e^{i\frac{2\pi}{Na}(N+n)qa} |q\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_q e^{i\frac{2\pi}{Na} nqa} |q\rangle$$

$$= |u_k\rangle$$

$\Rightarrow$ on se restreint à un intervalle de longueur N pour choisir n

$\Rightarrow$ on pose $-\frac{N}{2} < n < \frac{N}{2}$

4

2.4

$$H|u_k\rangle = \frac{1}{\sqrt{N}} \sum e^{ikqa} H|q\rangle \qquad E' \doteq E_0 + g\mu_B B + J$$

$$= \frac{1}{\sqrt{N}} \sum e^{ikqa} \left(E'|q\rangle - \frac{J}{2}\left(|q-1\rangle + |q+1\rangle\right)\right)$$

$$= \frac{1}{\sqrt{N}} \Big(E' \sum e^{ikqa} |q\rangle$$

$$- \frac{J}{2}\Big(e^{ika} \sum_{q'} e^{ikq'a} |q'\rangle$$

$$+ e^{-ika} \sum_{q'} e^{ikq'a} |q'\rangle \Big)$$

$$= E'|q\rangle - J\cos(ka) \cdot |q\rangle$$

$$= \left(E_0 + g\mu_B B + J(1-\cos ka)\right)|q\rangle$$

à $ka \ll 1$ $\quad \cos ka \simeq 1 - \frac{1}{2}(ka)^2$

$$\Rightarrow \quad E(k) \simeq \underbrace{E_0 + g\mu_B B}_{\text{offset}} + \underbrace{\frac{Ja^2}{2} k^2}_{\text{relation quadratique}}$$

→ sans importance

⇒ ondes de spin se comportent comme si elles avaient une masse

$$m = \frac{1}{Ja^2}$$

2.5 $S_z|u_k\rangle = \sum_{qq'} S_q^z \frac{1}{\sqrt{N}} e^{ikq'a} |q'\rangle$

$$\begin{cases} S_q^z|q'\rangle = \frac{1}{2}|q'\rangle \text{ si } & q \neq q' \\ S_q^z|q'\rangle = -\frac{1}{2}|q'\rangle \text{ si } & q = q' \end{cases}$$

$\hookrightarrow \quad S_q^z|q'\rangle = 1/2 - \delta_{qq'}$

$$\Rightarrow S_z|u_k\rangle = \frac{1}{\sqrt{N}} \sum_{qq'} e^{ikq'a} \left(\frac{1}{2} - \delta_{qq'}\right)|q'\rangle$$

$$= \frac{1}{\sqrt{N}} \Big(\frac{1}{2} N \sum_{q'} e^{ikq'a} |q'\rangle$$

$$- \sum_{q} e^{ikqa} |q\rangle \Big)$$

$$= \frac{N}{2}|u_k\rangle - 1 \cdot |u_k\rangle$$

2.6

$$P_q = |\langle q | u_k \rangle|^2$$
$$= \frac{1}{N} \left| \sum_{q'} e^{-iq'ka} \langle q | q' \rangle \right|^2$$
$$= \frac{1}{N} \left| e^{-iqka} \right|^2 = \frac{1}{N}$$

2.7

$$\langle u_k | S_q^\perp \cdot S_{q'}^\perp | u_k \rangle = \frac{1}{2N} \sum_{q''q'''} e^{-ik(q''-q''')a} \langle q'' | (S_q^- S_{q'}^+ + S_q^+ S_{q'}^-) | q''' \rangle$$
$$= \frac{1}{2N} \sum_{q''q'''} e^{ik(q'''-q'')a} \langle q'' | \left(\delta_{q'q'''} | q \rangle + \delta_{qq'''} | q' \rangle \right)$$
$$= \frac{1}{2N} \sum_{q''} \langle q'' | \left(e^{ik(q'-q'')a} | q \rangle + e^{ik(q-q'')} | q' \rangle \right)$$
$$= \frac{1}{2N} \left(e^{ik(q'-q)a} + e^{ik(q-q')a} \right)$$
$$= \frac{1}{N} \cos k(q'-q)a$$

3.1 $\qquad \mathcal{E}(k) = A k^2$

N = nbr d'états

$$\frac{4\pi}{3} k^3 = \left(\frac{2\pi}{L}\right)^3 N$$

$$N = \left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{3} k^3 = \left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{3} \left(\frac{\mathcal{E}}{A}\right)^{3/2}$$

$$\rho(\mathcal{E}) = \frac{dN}{d\mathcal{E}} = \underbrace{\left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{3} \frac{1}{A^{3/2}} \frac{3}{2}}_{} \cdot \sqrt{\mathcal{E}}$$

3.2 $\qquad \mu = 0$

$$\langle n \rangle = \int dE\, \rho(E) \frac{1}{e^{\beta E - \mu} - 1}$$

$$= \alpha \int \frac{dE \sqrt{E}}{e^{\beta E} - 1} = \alpha (k_B T)^{3/2} \cdot \underbrace{\int_0^{+\infty} \frac{\sqrt{u}\, du}{e^u - 1}}_{I}$$

$$= \left(\frac{L}{2\pi}\right)^3 \frac{2\pi}{A^{3/2}} I\, k_B^{3/2} T^{3/2}$$

3.3

$$M(T) = M(0) - \frac{M(0)}{N} \cdot \langle n \rangle$$

$$= M(0)\left(1 - \underbrace{\frac{M(0)}{N} \cdot \alpha'}_{\alpha} T^{3/2}\right)$$

3.4

$$U = \int dE\, E \, \frac{\rho(E)}{e^{\beta E} - 1} \propto T^{5/2}$$

$$\Rightarrow \quad C = \frac{dU}{dT} \propto T^{3/2}$$