Topological entropy and Arnold complexity for two-dimensional mappings

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Abstract

To test a possible relation between topological entropy and Arnold complexity, and to provide a nontrivial examples of rational dynamical zeta functions, we introduce a two-parameter family of discrete birational mappings of two complex variables. We conjecture rational expressions with integer coefficients for the number of fixed points and degree generating functions. We then deduce equal algebraic values for the complexity growth and for the exponential of the topological entropy. We also explain a semi-numerical method which supports these conjectures and localizes the integrable cases. We briefly discuss the adaptation of these results to the analysis of the same birational mapping seen as a mapping of two real variables. © 1999 Elsevier Science B.V. All rights reserved.

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Chaos is associated with extreme complexity, instability and ‘unpredictability’. Very few exact results are therefore known; however several exponents have been introduced to provide measures of the complexity and further, to classify chaotic systems [1,2]. The most popular are the Lyapounov exponents, which have a clear and intuitive interpretation, but require the choice of a metric and of an invariant measure. They can vary considerably under a very tiny change of the parameters of the dynamical system (see [2] p. 237): probing fine details, they are not universal. The same remarks also apply to the metric entropy [3,4]. By contrast there also exist exponents which do not involve any assumption about the phase space. These exponents, obviously, give a less detailed description of the system, but are more universal. They provide a means for a classification of dynamical systems. The topological entropy [5] log $h$ and the Arnold complexity [6] are two examples. The topological entropy probes the growth of the number of periodic orbits as a function of their
length, and the Arnold complexity, in two dimensions, probes the growth of the number of intersections of a straight line with its successive iterates. The Arnold complexity, $A_n$, is the number of intersections of a straight line with its $n$-th iterate. Generically, the Arnold complexity grows exponentially like $A_n = \lambda^n$. From now one we will consider, instead of the Arnold complexity $A_n$, $\lambda$ which characterizes asymptotically this exponential growth. The topological entropy and the Arnold complexity give informations not sensitive to specific details. From an intuitive point of view one can understand the importance of the topological entropy, since the asymptotic behavior of a dynamical system heavily depends on its fixed points and periodic orbits, hyperbolic fixed points playing a crucial role for chaos.

More precisely, we introduce here, for a mapping of two complex variables $k$, the fixed point generating function $H(t) = \sum h_n t^n$ where $h_n$ is the number of real, or complex, fixed points of the $n$-th power $k^n$ of the mapping. The same information is also coded in the so-called dynamical zeta function $\zeta(t)$ [7,8] introduced by Artin and Mazur [9] and related to the generating function $H(t)$ by $H(t) = t \frac{\zeta'(t)}{\zeta(t)}$. Both functions only depend on the number of fixed points, and not on their particular properties or localization: they are invariant under topological conjugacy (see Smale [10] for this notion). They do not depend on a specific choice of variables. $h$, the exponential of the topological entropy characterizes, how the coefficients $h_n$ grow with $n$. If, as we claim in this letter for the mappings we introduce below, $H(t)$ is rational then $h_n \sim \lambda^n$ where $\lambda$ is the modulus of the inverse of the smallest modulus pole of $H(t)$. Dynamical zeta functions and topological entropies have been calculated for some dynamical systems. This is, for instance, the case for explicit diffeomorphisms of the torus like the cat map [12], the three disks scattering [11] and also piecewise monotonic maps of the unit interval [13]. The topological entropy $h$ has also been calculated for ‘unpruned’ symbolic dynamics, $h$ being an integer [11]. We are not aware of any mapping of two complex variables with a noninteger value for $h$, other than the ones we present here. In this letter we will provide such an example of a discrete dynamical system with a rational dynamical zeta function, and, consequently, an algebraic value for the exponential of the topological entropy. This result should not be considered as a mathematical curiosity: it is similar to the rationality of critical exponents in conformal field theory. The algebraicity is the sign of deeper ‘rigid’ structures (like Feigenbaum cascades [14] are).

In the context of rational mappings it is easy to see that the Arnold complexity can be replaced by the complexity growth of the successive iterations. To define the complexity growth $\lambda$ we introduce the complexity generating function $G(t) = \sum d_n t^n$ where $d_n$ is the degree of any of the numerators, or denominators, of the components of the successive iterates of the rational mapping under consideration. When common polynomials factorize in the numerators and denominators, the coefficients $d_n$ grow more slowly than expected [15–17]. We stress that this definition only apply to rational mappings. The complexity growth $\lambda$ characterizes how coefficients $d_n$ grow with $n$: $d_n \sim \lambda^n$. Like $h$, the exponential of the topological entropy, the complexity $\lambda$ is the modulus of the inverse of the smallest modulus pole of $G(t)$. In this letter we claim, at least for the specific non-trivial birational example we consider, that the complexity growth $\lambda$ and $h$, the exponential of the topological entropy, are equal $\lambda = h$.

This will be tested successfully for a particular class of mappings, for which both generating functions are conjectured to be rational and, consequently, the complexity growth and the exponential of the topological entropy are algebraic. We will also give an effective semi-numerical method to compute these two characteristic numbers.

Let us introduce the discrete rational mapping $k_{\alpha, \epsilon}$ which associates $(u_{n+1}, v_{n+1})$ to $(u_n, v_n)$

$$
\begin{align*}
    u_{n+1} &= 1 - u_n + u_n / v_n, \\
    v_{n+1} &= \epsilon + v_n - v_n / u_n + \alpha (1 - u_n + u_n / v_n).
\end{align*}
$$

(2)

This mapping originates from the study of the symmetries of models of lattice statistical mechanics [18]. Depending on the actual values of the parameters $\alpha$ and $\epsilon$, the mapping can have completely different behaviors. For example, for $\epsilon = 0$ and whatever $\alpha$, as well as for $\alpha = 0$ and $\epsilon = -1$, 0, 1/2, 1/3 or 1, the mapping is integrable, whereas for all other values it is not [19]. A simple calculation shows that $k_{\alpha, \epsilon}$ is invertible and that its inverse
is also rational. This property of birationality is of important in our study. We have formally calculated the successive powers of \( k_{a,e} \) for arbitrary \( \alpha \) and \( \epsilon \), from which we propose

\[
G_{a,e}(t) = \frac{(1 + t)^2}{1 - t - 2 t^2 - t^3}.
\]

The expansion of the conjectured expression Eq. (3) coincides with our results up to the largest power \( n = 7 \) we were able to compute. Another rational expression with the same denominator is also obtained if one uses a matricial representation of the mapping Eq. (2) [20]. The expression Eq. (3) of \( G_{a,e} \) yields \( \lambda = 2.14789 \). To support this conjecture, we have used a semi-numerical method to estimate the complexity \( \lambda \). It consists in iterating \( k_{a,e} \) over the field of rationals. During the first steps, some ‘accidental’ cancellations between numerators and denominators can arise, but after this transient regime, the numerators and denominators become extremely large, and cancellations are only due to formal simplifications. We then determine how the magnitude of the four numerators, or denominators, grows with \( n \). With this method it is possible to raise \( k_{a,e} \) to the 15-th power, and moreover it is easy to scan a large number of values of the parameters \( \alpha \) and \( \epsilon \). The calculations are performed with an ‘infinite’ precision C-library [21]. Obviously, this method works only for rational mappings. On Fig. 1, one clearly sees that, for most of the values of \( \epsilon \), the complexity \( \lambda \) is extremely close to the expected value. We call ‘specific’ the values of the parameters for which the complexity \( \lambda \) is different from 2.14789, they will be discussed later. We also have formally computed for arbitrary \( \alpha \) and \( \epsilon \) the fixed points of the powers of \( k_{a,e} \) using, once again, the rationality of the mapping. This is explained in details in [22]. This gives

\[
H_{a,e}(t) = 2 t + 2 t^2 + 11 t^3 + 18 t^4 + 47 t^5 + 95 t^6 + 212 t^7 + \cdots.
\]

From this expression we propose the rational expression for the generating function of the number of fixed points \( k_{a,e} \)

\[
H_{a,e}(t) = \frac{(2 + 3 t^2 + t^3) t}{(1 - t^2)(1 - 2 t^2 - t^4)},
\]

or, equivalently, the dynamical zeta function reads

\[
\zeta_{a,e}(t) = \frac{(1 + t)(1 - t^2)}{1 - 2 t^2 - t^4}.
\]

Note that the total number of fixed points of \( k_{a,e}^n \) does not depend on the actual generic values of \( \alpha \) and \( \epsilon \), however the number of real fixed points is extremely dependent on these two parameters. Let us also mention a local area preserving property: the determinant of the Jacobian of the \( n \)-th power of \( k_{a,e} \) evaluated at each fixed points of \( k_{a,e}^n \) is equal to one. The ‘visual’ complexity of the phase diagram has its origin in the real fixed points, and therefore varies considerably with \( \alpha \) and \( \epsilon \) [23]. One sees that the two polynomials giving exponents \( \lambda \) and \( h \) are the same, and consequently we have the equality \( h = \lambda \). This equality holds for generic values of the parameters, however, as shown on Fig. 1, there exist nongeneric values. These nongeneric values include \( \epsilon = 1/3, 1/2, 3/5 \). It is then natural to investigate if the equality of the complexity growth and the exponential of the topological entropy also holds for the nongeneric values. We have performed calculations for these values and found that equality (1) is always true. The polynomials giving the value of \( \lambda \) and \( h \) are presented in Table 2. Probably, other nongeneric values exist, but they lead to simplifications occurring at very high orders, and the corresponding \( \lambda \) is too close to the generic value to be distinguished from it with our method.
Besides the specific values mentioned above, extra simplifications also happen for $\alpha = 0$, and the complexity is further reduced. We hence study this special case $\alpha = 0$. In that case, a change of variables [20], turns $k_{0,\epsilon}$ into a simpler mapping, $k_{\epsilon}$

$$y_{n+1} = z_n + 1 - \epsilon, \quad z_{n+1} = y_n \frac{z_n - \epsilon}{z_n + 1}. \quad (7)$$

From now on, the degrees, and the fixed points, are those of $k_{\epsilon}$. Since the complexity is lower, the semi-numerical method presented above is more efficient and it is possible to perform calculations beyond the 20th power. The results are displayed on Fig. 2, where the existence of integrable values, and nongeneric values, is clearly seen. It is simple to see that, if there is no simplification, $d_{n+1} = d_n + d_{n-1}$ where $d_n$ was introduced in the definition of generating function $G(t)$. In that case $G_k(t) = 2t - 1 = t(G(t) - 1) + t^2G_k(t)$. Up to the 20-th power there is no simplification and consequently we conjecture that, except for the specific values, the generating function of the complexity growth for $k_{\epsilon}$ is the following rational expression

$$G_k(t) = \frac{1 + t}{1 - t - t^2}. \quad (8)$$

The corresponding complexity growth is $\lambda = 1.61803$, in excellent agreement with Fig. 2. We have studied the possible equality between $h$ and $\lambda$ for the example $\epsilon = 13/25 = 0.52$. We have chosen this value, for which we present a detailed analysis, because it has no ‘accidental properties’ being neither of the form $1/m$ nor $\frac{n+1}{m+1}$ (see below and Fig. 2). We give in Table 1 the number of fixed points, as well as their properties. The corresponding phase portrait is very complicated and dominated by the real fixed points [23] which are all saddle or elliptic. We note that the same properties also holds for the complex fixed points. The expansion of $H_\epsilon$ can be deduced, up to order eleven, from Table 1. This expansion is compatible with the very simple rational form for the generating function of the number of fixed points for $k_{\epsilon}$

$$H_\epsilon(t) = \frac{(1 + t^2)t}{(1 - t^3)(1 - t - t^2)}, \quad (9)$$

or, equivalently, the dynamical zeta function is

$$\zeta(t) = \frac{1 - t^2}{1 - t - t^2}. \quad (10)$$

As expected, the two polynomials determining the exponential of the topological entropy and the complexity growth are equal, and so are $\lambda$ and $h$. Both are algebraic numbers.

Coming back to Fig. 2, we now analyze, for $\alpha = 0$, the specific values of $\epsilon$. Let us recall that $\epsilon = -1, 0, 1/3, 1/2, 1$ lead to integrable mappings [19]. This corresponds to a polynomial growth of complexity and of the number of fixed points, that is, $\lambda = h = 1$. This is seen on Fig. 2, except for $\epsilon = -1$, which is out of scale but for which this is also true. The other specific values are nonintegrable and can be partitioned in two sets: $(1/m; m > 3)$ and $((m - 1)/(m + 1); m > 3)$. In all cases the polynomials

![Fig. 2. The complexity growth $\lambda$ as a function of $\epsilon$ for $\alpha = 0$. $\epsilon$ is taken of the form $j/720$, the values $\alpha = 1/7, 1/11, 1/13, 5/7$ have also been added. The arrow indicates the conjectured generic value.](image)
giving the complexity growth and the exponential of the topological entropy are the same. The fact that this equality holds for an infinite family of values strongly support our conjecture. The polynomials are listed in Table 2.

In conclusion, we have given an example of a two-parameter family of two-dimensional discrete dynamical system with rational dynamical zeta function and rational degree generating function $G(t)$. For this example, $h$ is the exponent of the topological entropy and $\lambda$, which characterizes the Arnold complexity, have the same algebraic value. Moreover, when one restricts the parameter $\alpha$ to $\alpha = 0$ the value of $h = \lambda$ remains constant, except for an infinite discrete set of values of the parameter $\epsilon$. A semi-numerical method, applying to rational transformations only, has been given, which enables a calculation of the complexity growth, and the localization of possible integrable points. In fact the mapping considered here, belongs to a ‘very large’ family of transformations, for which similar results have also been obtained. This family of transformations is so large that (if one believes in ‘some’ universality of dynamical systems) most of the dynamical systems should be very closely ‘approximated’ by transformations having algebraic complexity values. Let us also mention that the phase diagrams of the transformation $k_{0,\alpha}$, for different values of $\epsilon$, are extremely different, even if the transformations have the same topological entropy and Arnold complexity. This is due to the fact that the phase diagrams represent two-dimensional pictures of the transformation considered as a transformation acting on real variables, whereas topological entropy and Arnold complexity amount to considering the transformation as acting on complex variables. Thus it is natural to slightly adapt the definitions of the previous notions, introducing a ‘real topological entropy’, and its exponential $\lambda_{\text{real}}$, which now correspond to the counting of the number of real fixed points, and a ‘real Arnold complexity’ which counts the number of real intersections of a generic real line with its $n$th iterate, and $\lambda_{\text{real}}$ which characterizes the exponential growth of this ‘real Arnold complexity’. A preliminary work shows, for our specific birational mapping, that one has, again, an equality $\lambda_{\text{real}} = \lambda_{\text{real}}$, for any values of the parameters, and, furthermore, that $h_{\text{real}}$ or $\lambda_{\text{real}}$, are still algebraic for many values of the parameters. Of course the corresponding ‘real’ generating functions do not have simple form like Eq. (8), or Eq. (10), anymore. We have also found that the common value of $h_{\text{real}}$ and $\lambda_{\text{real}}$, vary considerably with $\epsilon$. This is coherent with the large variety of phase diagram and this will be detailed in a forthcoming publication.

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References


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[21] The multi-precision library gmp is part of the GNU project.