Interface mapping in two-dimensional random lattice models

M Karsai¹,²,³, J-Ch Anglès d’Auriac²,⁴ and F Iglói¹,⁵

¹ Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary
² Institut Néel-MCBT CNRS⁶, BP 166, F-38042 Grenoble, France
³ Department of Biomedical Engineering and Computational Science, Aalto University Centre of Excellence in Computational Complex System Research, PO Box 12200 FI-00076 Aalto, Finland
⁴ Laboratoire de Physique Subatomique et de Cosmologie 53, avenue des Martyrs F-30026 Grenoble, France
⁵ Research Institute for Solid State Physics and Optics, H-1525 Budapest, PO Box 49, Hungary

E-mail: mkarsai@lce.hut.fi, dauriac@grenoble.cnrs.fr and igloi@szfki.hu

Received 2 June 2010
Accepted 6 August 2010
Published 31 August 2010

Online at stacks.iop.org/JSTAT/2010/P08027
doi:10.1088/1742-5468/2010/08/P08027

Abstract. We consider two disordered lattice models on the square lattice: on the medial lattice the random field Ising model at $T = 0$ and on the direct lattice the random bond Potts model in the large-$q$ limit at its transition point. The interface properties of the two models are known to be related by a mapping which is valid in the continuum approximation. Here we consider finite random samples with the same form of disorder for both models and calculate the respective equilibrium states exactly by using combinatorial optimization algorithms. We study the evolution of the interfaces with the strength of disorder and analyse and compare the interfaces of the two models in finite lattices.

Keywords: interfaces in random media (theory), self-affine roughness (theory), disordered systems (theory), optimization over networks

ArXiv ePrint: 1005.5666

---

⁶ UPR 5001 du CNRS, Laboratoire conventionné avec l’Université Joseph Fourier.
1. Introduction

The presence of quenched disorder of different types could influence the cooperative behaviour of many-body systems to different extents. In this respect dilution or randomness of ferromagnetic bonds, which is often called \( T_c \)-disorder, has a relatively weak effect. The ferromagnetic order in the pure system, which is present for \( T < T_c \), is not destroyed by a small amount of \( T_c \)-disorder. Dilution has a dramatic effect only in the vicinity of a phase transition point, \( |T - T_c|/T_c \ll 1 \). If this transition is of second order even its universality class can be changed for any small amount of disorder [1].

A disorder perturbation with a stronger effect is yielded by random fields [2]. As an example we consider here the random field Ising model (RFIM) in which the fields are taken from a distribution with zero mean and variance \( h_0^2 \). For dimension \( d > 2 \), the phase transition is governed by a zero-temperature fixed point, the ground state of the system being disordered for \( h_0 > h_{0,c}(d) > 0 \), and there is magnetic long-range order for \( h_0 < h_{0,c}(d) \). In two dimensions we have \( h_{0,c}(2) = 0 \); thus the magnetic order in the ground state is destroyed by any finite random field [3, 4]. The ground state of the RFIM is conveniently characterized by geometric (or Ising) clusters [5], which are formed by adjacent spins being in the same orientation. In 2D these clusters are homogeneous only up to a finite characteristic length, \( \ell_g \), which is called the breaking-up length of geometrical clusters [6]–[8]. In a coarse-grained picture the ground state consists of randomly oriented domains of typical size \( \ell_g \), and as a consequence there is no long-range order in the system. For weak disorder, \( h_0 \ll J \), \( J = 1 \) being the nearest neighbour coupling constant, the
breaking-up length is large, and for $h_0 \to 0$ it diverges as [6]:

$$\ell_d(h_0) \sim \exp(Ah_0^{-2}),$$

with a disorder dependent constant, $A = O(1)$.

As far as first-order transitions are concerned the effect of bond disorder is also strong and the physical phenomenon is analogous to that of the RFIM discussed before [9]. As a prototypical model having a first-order phase transition we consider the $q$-state Potts model [10] in which the number of states is sufficiently large, $q > q_c(d)$. In the pure system having homogeneous couplings ($J = J_0$) at the first-order transition point, $T = T_c$, the ordered and disordered phases coexist, the latent heat is finite and there is a finite jump of the order parameter. Introducing random ferromagnetic couplings, such that the fluctuating part, $\kappa = (J - J_0)/J_0$, has zero mean and variance $\kappa_0^2$, the latent heat and the jump of the magnetization are generally reduced with increasing $\kappa_0$. For dimension $d > 2$ for weak disorder, $\kappa_0, c(d) > \kappa_0 > 0$, the transition stays first order, whereas for strong disorder, $\kappa_0 > \kappa_0, c(d)$, the latent heat is vanishing and the transition softens to second order [4,11]. The critical exponents of this transition are independent of the strength of the disorder, but they generally depend on the value of $q$ [12–14], [16,17]. In the borderline case, $\kappa_0 = \kappa_0, c(d)$, there is a tricritical point with a new class of tricritical exponents, which are expected to be independent of the value of $q$, and the actual fixed point is at $q \to \infty$ [18]. In 2D the limiting value is $\kappa_0, c(2) = 0$ and the phase transition is softened to second order by any weak bond disorder [4,11].

To characterize the state of the system we consider the high-temperature series expansion [19], which in the large-$q$ limit is dominated by a single diagram. In the pure case this diagram is either the fully connected graph, which happens in the ordered phase, $T < T_c$, or the fully disconnected graph, which is the case in the disordered phase, $T > T_c$. At the phase transition point these two basic graphs coexist. In the disordered model the dominant diagram at $T = T_c$ contains both connected and disconnected parts. In 2D the optimal diagram is homogeneous only up to a characteristic finite length, $\ell_d$, which is the breaking-up length of the optimal diagram. For small disorder the breaking-up length is large and in the limit $\kappa_0 \to 0$ it is divergent [18]:

$$\ell_d(\kappa_0) \sim \exp(B\kappa_0^{-2}),$$

which is in the same form as for the RFIM in equation (1). In the thermodynamic limit the optimal diagram is a random composition of the two basic graphs of typical size $\ell_d$; therefore there is no phase coexistence in 2D and the transition is of second order.

As described above, the RFIM and the random bond Potts model (RBPM) have analogous physical properties. This analogy was first noticed by Cardy and Jacobsen [13] (CJ) and they have obtained an exact mapping, which is valid in the limit of $d \to 2$ and $q \to \infty$, when the interface Hamiltonians of the two problems have the same type of solid-on-solid (SOS) description. From this mapping it follows that the energy exponents of the tricritical RBPM are equivalent to the magnetization exponents of the critical RFIM. Furthermore, analysing the renormalization group (RG) flow it was conjectured that the above mapping stays valid for $d > 2$, in particular at $d = 3$. This conjecture has been numerically tested [18] for the 3D RBPM in the large-$q$ limit.

The CJ mapping has consequences at $d = 2$, as far as interface properties of the two models are concerned at large scales. This has already been mentioned with respect to
the same form of the breaking-up lengths in equations (1) and (2). In this context one may ask the question of how this mapping works on finite scales. In this paper we are going to perform a systematic study and consider both models with exactly the same form of disorder. For this we put the Potts model on the square lattice, whereas the Ising model is on the medial lattice. In this way, couplings of the Potts model and fields of the Ising model are in the same position. We study random samples with fixed boundary conditions, which favour the opposite orientation for the RFIM and the different phases (basic graphs) for the RBPM for the two halves of the lattice. Using a combinatorial optimization algorithm [20, 21] we calculate exactly the ground state of the RFIM and the optimal diagram of the RBPM and study their evolution with increasing strength of disorder. We also compare the diagrams obtained in the two problems and study their fitting.

The structure of the rest of the paper is the following. The RFIM and the RBPM are defined in section 2 together with the asymptotic mapping. Evolution of the diagrams with the strength of disorder is presented in section 3 whereas a comparison of the diagrams for the two models is given in section 4. Our paper is closed with a discussion in section 5.

2. Models and asymptotic mapping

We consider a square lattice whose sites are at \((x, y)\) with \(1 \leq x \leq L\) and \(- (L - 1)/2 \leq y \leq (L - 1)/2\), both \(x\) and \(y\) being integer. In the corresponding medial lattice the sites are placed at the middle points of the links and have coordinates \((u, v)\) as \((x, y + 1/2)\) and \((x + 1/2, y)\); see figure 1.
2.1. The random field Ising model

Spins of the random field Ising model, \( \sigma_{u,v} = \pm 1 \), are put on the vertices of the medial lattice. The Hamiltonian of the model is given by

\[
\mathcal{H}_I = - \sum_{u,v} \sigma_{u,v} \sigma_{u+1/2,v+1/2} - \sum_{u,v} h(u,v) \sigma_{u,v}.
\]

The external field, \( h(u,v) \), is a random number which is parametrized as

\[
h(u,v) = h_0 \epsilon(u,v),
\]

where the distribution of \( \epsilon(u,v) \) has zero mean and variance unity. We apply fixed spin boundary conditions: the Ising spins for the upper part of the boundary (\( v > 0 \)) are fixed at +1, and for the lower part of the boundary (\( v < 0 \)) they are fixed at -1. This model is studied at \( T = 0 \). For a given realization of the disorder we calculate the ground state exactly by using a combinatorial optimization algorithm.

2.1.1. The SOS approximation. For weak disorder, such that \( \ell_g(h_0) > L \), the ground state is well approximated by two clusters with +1 and -1 spins, respectively, and having an interface in between. Furthermore we use the solid-on-solid (SOS) approximation, where the height of the interface at position \( u \) is given by a unique function, \( \zeta(u) \). In this case the position of the interface is obtained by minimizing the SOS energy functional [13]:

\[
E(\zeta) = - \sum_u h_0 \left( \sum_{v < \zeta(u)} - \sum_{v > \zeta(u)} \right) \epsilon(u,v),
\]

with the constraint that the length of the interface measured in the medial lattice is constant.

2.2. The random bond Potts model

The random bond Potts model is put on the vertices of the original square lattice and defined by the Hamiltonian

\[
\mathcal{H}_P = - \sum_{x,y} [J(x, y + 1/2) \delta(s_{x,y}, s_{x,y+1}) + J(x + 1/2, y) \delta(s_{x,y}, s_{x+1,y})].
\]

Here \( s_{x,y} = 1, 2, \ldots, q \) is a Potts spin variable at site \( (x, y) \) and the couplings, \( J(x, y + 1/2) \equiv J(u,v) > 0 \) and \( J(x + 1/2, y) \equiv J(u,v) > 0 \), are independent and identically distributed random numbers. Introducing the reduced temperature, \( T' = T \ln q \), and similarly, \( \beta' = 1/T' = \beta / \ln q \), the partition function in the random cluster representation is given by [19]

\[
Z = \sum_{G \subseteq E} q^{c(G)} \prod_{(u,v) \in G} [q^{K(u,v)} - 1]
\]

with \( K(u,v) = \beta' J(u,v) \). Here the sum runs over all subsets of bonds, \( G \subseteq E \), and \( c(G) \) stands for the number of connected components of \( G \). In the following we consider the
large-\(q\) limit, where \(q^K(u,v) \gg 1\), and the partition function can be written as

\[
Z = \sum_{G \subseteq E} q^{\phi(G)}, \quad \phi(G) = c(G) + \sum_{(u,v) \in G} K(u,v)
\]

which is dominated by the largest term, \(\phi^* = \max_G \phi(G)\), so \([16, 22]\)

\[
Z = q^{\phi^*} + \text{subleading terms}. \tag{9}
\]

For random couplings with \(0 < K(u,v) < 1\) we use the form \(K(u,v) = K(1 + \kappa(u,v))\) and the \(\kappa(u,v)\) random numbers are taken from a symmetric distribution: \(P(\kappa) = P(-\kappa)\), with a variance \(\kappa_0^2\); thus \(\kappa(u,v) = \kappa_0 \epsilon(u,v)\). For this type of distribution the phase transition point of the system follows from self-duality and is given by \([23]\)

\[
K_c = \frac{1}{2}. \tag{10}
\]

Thus at the critical point the random couplings are parametrized as

\[
\beta'J(u,v) = K(u,v) = \frac{1}{2}[1 + \kappa_0 \epsilon(u,v)] \tag{11}
\]

and the strength of disorder is measured by \(\kappa_0\). In our problem we use such boundary conditions that the boundary couplings for the upper part of the lattice \((v > 0)\) are strong \(\kappa^+ = 1\), thus promoting an ordered phase, whereas for the lower part of the lattice \((v < 0)\) these are weak, \(\kappa^- = 0\), thus favouring the disordered phase. For a given realization of the disorder we calculate the ground state exactly by using a combinatorial optimization algorithm \([21]\).

2.2.1. The SOS approximation. In the following we consider the model at the critical point in the weak disorder limit, when \(L_d(\kappa_0) > L\). In this case we use the approximation that the optimal diagram consists of a fully connected part, denoted by \(G_c\), and a fully disconnected part, denoted by \(\overline{G_c}\), having an interface in between. The interface is approximated with the solid-on-solid (SOS) model, having a unique height \(\zeta(u)\) at position \(u\) \([13, 17]\). The lattice contains \(N = L^2\) points, from which \(n(G_c)\) and \(n(\overline{G_c})\) are in \(G_c\) and in \(\overline{G_c}\), respectively. Similarly, out of the \(E = 2L(L-1)\) edges, there are \(e(G_c)\) in the subgraph \(G_c\). The cost function in equation (8) is given by

\[
\phi = N + 1 - n(G_c) + \sum_{(u,v) \in G} (1 + \kappa_0 \epsilon(u,v))/2
\]

\[
= N + 1 - n(G_c) - e(G_c)/2 + \sum_{(u,v) \in G} \kappa_0 \epsilon(u,v)/2. \tag{12}
\]

Now we use the microcanonical condition \(\sum_{\text{all bonds}} \epsilon(u,v) = 0\) and arrive at the cost function:

\[
4\phi(\zeta) = \kappa_0 \sum_u \left( \sum_{v < \zeta(u)} - \sum_{v > \zeta(u)} \right) \epsilon(u,v), \tag{13}
\]

for a fixed value of \(n(G_c) - e(G_c)/2\). This latter quantity is uniquely determined by the length of the interface, which is measured in the square lattice.

Comparing the two expressions in equations (5) and (13) we see that in the SOS approximation the position of the interface is obtained by finding the extremal value of
the same expression, however with somewhat different constraints. In both cases the length of the interface is fixed; however in the RFIM this length is measured in the medial lattice, whereas for the RBPM it is measured in the original lattice. In the following we study both models numerically and see how well the SOS approximation and, thus, the above relation remain valid in finite systems.

### 3. Evolution of interfaces with the strength of disorder

In the numerical study we use bimodal disorder, such that \( \epsilon(u, v) = \pm 1 \) with the same probability. The numbers of samples used in the calculation are shown in Table 1. The limitations in the calculations are due to the RBPM, for which the combinatorial optimization algorithm needs more computational time.

#### 3.1. The random field Ising model

For the RFIM the evolution of the ground state with the strength of disorder, \( h_0 \), for a given realization of the random numbers, \( \epsilon(u, v) \), is illustrated in figure 2. In the limit \( h_0 \to 0^+ \) there is a unique SOS interface between two oppositely magnetized clusters. With increasing \( h_0 \) at \( h_{0,1} \) the ground state will change. It still consists of two large clusters; however the interface has some overhangs, and thus it is no longer of SOS type. With further increased disorder, for \( h_0 \geq h_{0,2} \), it is no longer the case that two clusters exist, but some islands are formed in the big clusters. Finally, for strong disorder, \( h > h_{0,3} \), the ground state is represented by several disjoint clusters and one cannot identify an interface in the system.

We have studied the behaviour of the first relevant disorder scale, \( h_{0,1} \), at which the SOS picture is broken down. Having several samples we have calculated its mean value, which is then repeated for different finite sizes, \( L \). \( \bar{h}_{0,1}(L) \) is found to go to zero with \( L \) as a power law, \( \bar{h}_{0,1}(L) \sim L^{-\omega} \), which is illustrated in figure 3. The exponent is found to be \( \omega = 0.36(5) \). Similarly, the second disorder scale, \( h_{0,2} \), is found to follow the same type of decay with \( L \), with the same exponent, \( \omega \), but with a different prefactor.

Our numerical results are in agreement with the exact result, that for any finite \( h_0 > 0 \) in the thermodynamic limit there is no long-range order in the 2D RFIM.

#### 3.2. The random bond Potts model

We have repeated the previous analysis for the RBPM. For a given random sample we have varied the strength of disorder, \( \kappa_0 \), and studied the form of the optimal diagram. Having the same set of \( \epsilon(u, v) \) as for the RFIM in figure 2 the results are collected in

---

**Table 1.** Number of samples, \( N \), used in the calculation for different finite systems of linear size, \( L \).

<table>
<thead>
<tr>
<th>L</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>405</td>
<td>146</td>
<td>71</td>
<td>13</td>
</tr>
</tbody>
</table>

---

7 This point should be related to the percolation transition of the geometrical clusters [7, 8]. For \( h < h_{0,3} \) there is an infinite geometrical cluster in the system, whereas for stronger random fields there is no spanning cluster.
Figure 2. Evolution of the ground state of the RFIM with increasing strength of disorder in a $32 \times 32$ lattice. Up and down spins are represented by dark (red) and light (green) regions. (a) $h_0 = 0.1275$: two clusters with an SOS interface. (b) $h_0 = 0.2525$: two clusters with an interface with overhangs. (c) $h_0 = 0.3275$: internal islands are formed within the big clusters. (d) $h_0 = 4.0$: several disjoint clusters are formed.

Figure 3. Average value of the critical disorder strength for the RFIM, $h_{0.1}$, and that for the RBPM, $\kappa_{0.1}$, as a function of $L$ on a log–log scale. The straight line indicating the theoretical prediction has a slope $-\omega = -1/3$. 

doi:10.1088/1742-5468/2010/08/P08027
Interface mapping in two-dimensional random lattice models

Figure 4. Evolution of the optimal set of the RBPM with increasing strength of disorder in a $32 \times 32$ lattice having the same set of $\epsilon(u,v)$ as for the RFIM in figure 2. Connected graphs and isolated sites are represented by dark (red) and light (green) regions. (a) $\kappa_0 = 0.25$: two clusters with an SOS interface. (b) $\kappa_0 = 0.34$: two clusters with an interface with overhangs. (c) $\kappa_0 = 0.5$: internal islands are formed within the big clusters. (d) $\kappa_0 = 0.76$: several disjoint clusters are formed.

In the limit $\kappa_0 \to 0$ the interface is a plane, with a minimal length, $L_{\text{int}} = L$ (not shown in figure 4). At a limiting value, $\kappa_{0,0}$, the interface becomes an SOS interface with $L_{\text{int}} > L$. On further increasing $\kappa_0$ the length of the interface is increasing, but it stays of SOS type. Afterwards, at $\kappa_{0,1}$ the interface will lose its SOS character: overhangs and bubbles appear. Finally, for strong disorder the optimal diagram is composed of disjoint subgraphs. We can thus conclude that the evolution of the interfaces in the RBPM and in the RFIM with the strength of disorder are qualitatively very similar. There are, however, differences, for $\kappa_0 \leq \kappa_{0,0}$, which are due to the different orientations of the interface lines.

We have also studied the size dependence of the average value of $\kappa_{0,1}$, which is shown in figure 3. Here also a power-law decay is found with the same exponent, $\omega$, as for the RFIM.

We can explain the value of the $\omega$ exponent, which is measured both for the RFIM and for the RBPM, in the following way. Let us concentrate now on the RFIM; for the RBPM similar reasoning works. At $h_{0,1}$ the first overhang in the ground state of the RFIM appears. The difference between the ground states for $h < h_{0,1}$ and $h > h_{0,1}$ has a narrow shape; see figure 5. Its typical width is $w = O(1)$, since the total length of the interface is increased by $\Delta L_{\text{int}} = O(1)$, and its height is given by $L_\perp$, which is the size of the transverse fluctuations of the SOS interface and given by $L_\perp \sim L^{2/3}$. Now we

doi:10.1088/1742-5468/2010/08/P08027
Figure 5. Schematic plot of the variation of the interface of the RFIM with increasing $h$. For $h < h_{0,1}$ the interface has an SOS character (the $-$ spins are in the lower (red) domain) whereas for $h > h_{0,1}$ there is an overhang in the interface and the $-$ spins gain an extra narrow shaded (blue) region. This narrow region has a width of $w = O(1)$ and a length of the size of the transverse fluctuations: $L_{\perp} \sim L^{2/3}$.

use the Imry–Ma argument \cite{3} and compare the loss of energy due to the increase of the interface: $E_{\text{int}} \sim w$, with that of the gain due to disorder fluctuations: $E_{\text{fl}} \sim h_{0,1} \sqrt{wL_{\perp}}$. From the condition $E_{\text{int}} = E_{\text{fl}}$ we obtain the estimate $h_{0,1} \sim L^{-1/3}$; thus the exponent is given by $\omega = 1/3$ in agreement with the numerical results.

4. Comparison of the structures of the graphs for the two models

The SOS mappings presented in section 2 yield that for weak disorder the scaling behaviours of the interfaces in the two models are asymptotically equivalent. Here we study the question of what happens for finite systems and for not too weak disorder, and which type of similarity occurs between the ground states of the two problems. For this study we consider exactly the same random samples for the two models, which means that the sets of random variables, $\epsilon(u,v)$, are the same for each position, $(u,v)$; just the strengths of the disorder, $h_0$ and $\kappa_0$, respectively, can be different. For a given sample we calculate the ground state of the RFIM, as well as the optimal set for the RBPM, and compare them. In order to define a quantitative measure of the difference between the two graphs we consider the medial lattice and assign to each site, $(u,v)$, a variable denoted by $D(u,v)$. If in the ground state of the RFIM $\sigma(u,v) = +1$ ($\sigma(u,v) = -1$) and at the same time in the optimal set of the RBPM the edge $(u,v)$ is occupied (non-occupied), then $D(u,v) = 0$; otherwise $D(u,v) = 1$. In figure 6 we compare the ground state configurations in figure 2 with the optimal sets in figure 4. One can see in this figure that close to the boundaries $D(u,v)$ is typically zero (greyscale (gsc) 3 (red) and gsc 2 (green) points) and in the interface region we have sites with $D(u,v) = 1$ (gsc 1 (yellow) points).
Figure 6. Comparison of the ground state of the RFIM with the optimal set of the RBPM for the same pair of disorder parameters \((h_0, \kappa_0)\) as are given in the panels of figures 2 and 4, respectively. Greyscale (gsc) 3 (red) points represent: up spins in the RFIM and the connected graph in the RBPM; gsc 2 (green) points: down spins and isolated sites; gsc 1 (yellow) points are for non-coherent sites with \(D(u, v) = 1\). The values of the discrepancies are \(\delta = 24/1984, 135/1984, 187/1984\) and \(559/1984\), for panels (a), (b), (c) and (d), respectively.

The difference between the two graphs is defined by the fraction of non-coherent sites:

\[
\delta(h_0, \kappa_0) = \frac{1}{2L(L - 1)} \sum_{(u, v)} D(u, v),
\]

which is what we call the discrepancy. Here \(2L(L - 1)\) is the number of sites in the medial lattice with the given boundary condition.

In the actual calculation we have fixed the value of the RBPM disorder parameter, \(\kappa_0\), and calculated the discrepancies for different values of \(h_0\). In this way we have measured the minimum value of the discrepancy, \(\delta_{\text{min}}(\kappa_0)\), and the corresponding value of the RFIM parameter\(^8\), \(\tilde{h}_0\). The average value of the minimal discrepancy, \([\delta_{\text{min}}]_{\text{av}}(\kappa_0)\), versus \(\kappa_0\) is plotted in figure 7 for different finite systems from \(L = 16\) to \(128\).

For large \(\kappa_0\) the curves have a plateau\(^9\), then with decreasing \(\kappa_0\) they start to decrease, pass over a minimum and afterwards increase for small \(\kappa_0\). With increasing size the value

\(^8\) For a given value of \(\kappa_0\) there is a finite interval of \(h_0\) which corresponds to the minimal discrepancy condition. Here we use the smallest such value as \(\tilde{h}_0\).

\(^9\) For large \(\kappa_0\) the discrepancy approaches a finite limiting value, to which there are finite-size corrections. Since in the vicinity of an ordered boundary the discrepancy is zero these corrections are expected to be in the form \(\sim - A/L\), in agreement with the numerical results in figure 7.
at the minimum \([\delta_{\text{min}}]_{\text{av}}(L)\) is decreasing and in the large-\(L\) limit the minimum is expected to be shifted at the origin. Indeed in the inset of figure 7 we have plotted \([\delta_{\text{min}}]_{\text{av}}(L)\) as a function of \(L\), and on a log–log scale an asymptotically linear dependence is found; thus \([\delta_{\text{min}}]_{\text{av}}(L) \sim L^{-\omega'}\). Here the exponent is \(\omega' = 0.39(7)\), which is compatible with \(\omega' = 1/3\), the value of which follows from the following simple argument. For weak disorder the interface in the RBPM is approximately flat (see the reasoning in section 3) whereas in the RFIM it is rough having a transverse fluctuation of size \(L_{\perp} \sim L^{2/3}\). In the interface region there are \(L \times L_{\perp}\) sites; consequently the average discrepancy behaves as \(\sim L^{-1/3}\), in agreement with the numerical results. We can thus conclude that in the large-\(L\) limit the behaviour of the discrepancy with the strength of disorder is the following. It starts from zero at \(\kappa_0 = 0\), has an approximately quadratic variation for small disorder and approaches a plateau for stronger disorder. This behaviour is in complete agreement with the SOS mapping in section 2.

We have also checked the actual forms of the graphs in the two problems for strong disorder. As illustrated in figure 6 the two graphs generally have quite similar structures, as far as the subgraphs, the topology of the islands, etc are concerned.

5. Discussion

In this paper we have considered two problems of lattice statistics in which disorder has strong and somewhat analogous effects as far as the cooperative behaviours of the models are concerned. In both problems geometrical interpretation of the state of the
system is used. In the first problem, which is the two-dimensional Ising model, we use clusters of parallel spins (geometrical clusters) to characterize the ground state. Here random fields destroy the ferromagnetic order at $T = 0$ and the geometrical clusters have homogeneous parts of finite extent, $\ell_g(h_0)$. In the second problem, which is the two-dimensional Potts model with a large number of states, the optimal diagram of the high-temperature series expansion is used as a geometrical interpretation. Here we consider the first-order transition point of the pure system, whose transition is softened to a continuous one due to bond disorder. As a consequence at the transition point there is no phase coexistence and the homogeneous parts of the optimal diagram have a finite extent of $\ell_d(\kappa_0)$.

In this paper we have considered finite samples of linear length $L$, varying the strengths of disorder, $h_0$ and $\kappa_0$, in the two problems, respectively. Having fixed spin boundary conditions for $\ell_g(h_0) > L$ ($\ell_d(\kappa_0) > L$) there are basically two phases (two elementary diagrams) in the ground state (optimal diagram) of the RFIM (RBPM), which are separated by an interface. For large scales and in the SOS approximation the interface Hamiltonians of the two problems have similar forms.

Here we have studied the validity of the interface mapping in finite systems, by putting the Potts model on the square lattice and the Ising model on the medial lattice. In this way the bonds of the Potts model and the fields of the Ising model have the same location. Using the same disordered samples we have studied the evolution of the interfaces in the two models as the strength of disorder is gradually increased. We have seen a similar trend of the evolution in the two models; however, for finite lattices we have also observed differences. These differences are basically due to the interfaces having different orientations in the two models: in the RBPM it is in the $(1, 0)$ direction, whereas in the RFIM it has a $(1, 1)$ orientation. In the SOS approximation the interfaces are obtained by minimizing the same cost function, however with different constraints in the two problems.

We have also compared the cluster structures of the two problems and have obtained the following conclusion. The relative difference between the two geometrical objects as quantified by the discrepancy in equation (14) tends to zero if (i) the size of the system goes to infinity and (ii) the strength of disorder goes to zero. The finite-size corrections are found to be in power-law form, and vanish as $L^{-\omega}$ with $\omega = 1/3$. Consequently relatively large finite samples are needed to see the asymptotic behaviour.

Acknowledgments

This work has been supported by the Hungarian National Research Fund under grant Nos OTKA K62588, K75324 and K77629. FI is indebted to the Institut Néel–MCBT for hospitality during the final stages of the work. MK thanks the Ministère Français des Affaires Étrangères for a research grant.

References


doi:10.1088/1742-5468/2010/08/P08027
Interface mapping in two-dimensional random lattice models

Chatelain C and Berche B, 1999 Phys. Rev. E 60 3853
Picco M, unpublished

doi:10.1088/1742-5468/2010/08/P08027