Petr Akhmet'ev and Olivier Cépas

A visualization of stable invariants of integer dynamics of three-colorings

Abstract

Winding loops in models with local constraints have a natural integer dynamics consisting in the evolution of their integer winding numbers. The dynamics in this case, known as Kempe moves, results in disconnected sectors. We show that the stable invariant I_2, introduced in [1], is visualized as right-left configurations of winding loops on the immersed Konstantinov torus. A possible connection of Kempe sectors with stable homotopy groups of spheres is discussed.

1. Preliminaries

1.1 Integer dynamics on the L-periodic lattice

The model consists of coloring the edges of a regular hexagonal lattice Λ of linear size L (and $N = 3L^2$ edges) with three colors, *e.g.* A, B, C, such that each vertex has three edges colored with three different colors, see [1] for details. The factorization of the lattice Λ by the two L-periods is the standard 2-torus, denoted by T^2. Because the lattice Λ is hexagonal, we have additional symmetries. Take a point o, a vertex of the lattice. The rotation $S : \Lambda \to \Lambda$ through o with angle $\frac{2\pi}{3}$, which is an invariant transformation of Λ, is well-defined. As well as take 3 lines l_x, l_y, l_z on the lattice with centre o, $S(l_x) = l_y$, $S(l_y) = l_z$, $S(l_z) = l_x$. Denote by $R_i : \Lambda \to \Lambda$ the reflection symmetry with respect to the line l_i, $i \in \{x, y, z\}$. The corresponding orientation-preserved diffeomorphism

$$S : T^2 \to T^2 \qquad (1)$$

of order 3, and the orientation-reversed reflection

$$R_i : T^2 \to T^2 \qquad (2)$$

of order 2 on the factor of the plane $\mathbb{R}^2$, $\Lambda \subset \mathbb{R}^2$, are well-defined.

The successive edges of two colors, say B and C (or A and B, or A and C) form self-avoiding closed loops. A Kempe move is an exchange of the two colors of one of these loops. It gives a new valid 3-coloring (all edges colored with three different colors). An equivalent relation called Kempe moves gives a partition of 3-colorings in equivalent classes.

The $A-B$, $B-C$, $C-A$ cycles are well-defined as oriented cycles. This gives the elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in H_1(T^2;\mathbb{Z}) \equiv \mathbb{Z}\oplus\mathbb{Z}$, where we denote $\mathbb{Z}\oplus\mathbb{Z}$, a direct sum of the group of integers. Recall that, in this case, we get an isomorphism between the fundamental group $\pi_1(T^2, pt)$ and the homology group $H_1(T^2;\mathbb{Z})$ with integer coefficients. In particular, the fundamental group does not depend on a marked point $pt \in T^2$ on the torus. Each element is an integer vector (a collection of two integers (n,m)) on the lattice $\mathbb{Z}\oplus\mathbb{Z} \subset \mathbb{R}\oplus\mathbb{R}$. A triplet $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ satisfies the equation

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0.$$

A Kempe move transforms a triplet $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ into $\{\mathbf{a}_1,\mathbf{b}_1,\mathbf{c}_1\}$ as follows. We can choose $\mathbf{a}_l = k\hat{\mathbf{a}}$, where $\hat{\mathbf{a}} = \frac{\mathbf{a}}{\gcd(a_x,a_y)}$, $\mathbf{a} = (a_x, a_y)$ with k a positive or negative integer:

$$\{\mathbf{a}_1;\mathbf{b}_1;\mathbf{c}_1\} = \{\mathbf{a}+2k\hat{\mathbf{a}}; \mathbf{b}-k\hat{\mathbf{a}}; \mathbf{c}-k\hat{\mathbf{a}}\}.$$

This transformation keeps the direction of the vector $\mathbf{a}$. The two analogous transformations with invariant directions of the vectors $\mathbf{b}$, $\mathbf{c}$ are also possible:

$$\{\mathbf{a}_1;\mathbf{b}_1;\mathbf{c}_1\} = \{\mathbf{a}-k\hat{\mathbf{b}}; \mathbf{b}+2k\hat{\mathbf{b}}; \mathbf{c}-k\hat{\mathbf{b}}),$$

$$\{\mathbf{a}_1;\mathbf{b}_1;\mathbf{c}_1\} = \{\mathbf{a}-k\hat{\mathbf{c}}; \mathbf{b}-k\hat{\mathbf{c}}; \mathbf{c}+2k\hat{\mathbf{c}}).$$

The problem to to classify equivalent classes of triplets $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ up to the transformations $\{\mathbf{a},\mathbf{b},\mathbf{c}\} \mapsto \{\mathbf{a}_1,\mathbf{b}_1,\mathbf{c}_1\}$. We are looking for finite-type invariants of the transformation.

There are the following list of invariants:

1. $\chi(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = a_x b_y - b_y a_x \pmod 2$.
2. For $\chi(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = 1$, define:

$$I_2(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = 2a_x b_y + 2a_y b_y + a_x b_y + a_y b_x \pmod 4, \qquad (3)$$

we get $I_2 = \pm 1 \pmod 4$ [1], the formula (39).

3. For $\chi(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = +1$ we define the invariant I^+ as follows. Denote $-a_x - a_y$ by a_z, for b_z, c_z the formula are analogous. Denote the 3-component extension of $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ by $\{\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}\}$. We may assume that the following equation (up to cyclic permutation of the vectors $\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}$) is satisfied:

$$\{\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}\} \pmod 2 = \{(1,0,1),(0,1,1),(1,1,0)\}. \tag{4}$$

Define, in the case $I_2(\{\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}\}) = +1$, [1] using the formula (57)

$$I^+(\{\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}\}) = \frac{1}{2}(\tilde{a}_y\tilde{b}_x - (\tilde{a}_y + \tilde{b}_x + \tilde{c}_z)) \pmod 4. \tag{5}$$

In the two other cyclic permutations, the invariant I^+ is similarly defined by the corresponding permutations of the vectors $\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}$. If $\{\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}\}$ (mod 2) are given by an odd permutation, we may use an analogous formula with a different sign of the linear terms. In the case $I_2(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = -1$ the formula for $I^-(\{\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}}\})$ is a little different, but analogous. The goal of the paper is the visualization of the formula (3), by using immersions.

1.2 Immersions

Let us consider the Konstantinov immersion [2], p. 434-437, figs. 2-6, which is denoted by $\varphi' : T^2 \looparrowright \mathbb{R}^3$. Let us describe the surface, using the figures. We start with the orthogonal projection $\pi \circ \varphi' : T^2 \to \mathbb{R}^2$, $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ on the hyperplane.

Let us consider two domains on fig. 2 and isometrically identify corresponding segments $[a,b]$ with $[a',b']$, $[c,d]$ with $[c',d']$ and $[e,f]$ with $[e',f']$. We get a surface with a circle boundary. This surface is homomorphic to a torus without a disk. Let us call this surface the first sheet of the torus. The first sheet is an immersed domain on the plane (the central hexagonal points are the double points of the domain): this domain is on figs. 3 and 4. The boundary circle is immersed on the plane as the central curve $\pi\Sigma_1$ with 6 self-intersection points. The second sheet of the torus is the cylinder, immersed on the plane, the boundary of this cylinder contains 2 circles. The interior circle coincides with $\pi\Sigma_1$, the exterior circle is the standard embedded circle $\pi\Sigma_2$. On fig. 3, this cylinder is cut into 3 disjoint squares along the segments $[KL]$, $[MN]$ and $[OP]$. The third sheet is a disk with boundary $\pi\Sigma_2$. The constructed surface is the image of the fold map $\pi\circ\varphi' : T^2 \mapsto \mathbb{R}^2$ with the fold (the outline of the projection) $\pi\Sigma_1 \cup \pi\Sigma_2$. The image of the projection of the Konstantinov immersion on the plane is non-chiral. Before to investigate the Konstantinov immersion φ', let us describe symmetries and cycles on this projection.

Cycles on the torus are realized as closed curves on the first sheet of the surface. We may introduce coordinates x, y, z in $H_1(T^2;\mathbb{Z})$ and cycles $l_x, l_y, l_z \in H_1(T^2;\mathbb{Z})$ (our notations correspond to notations of Subsection 1.1) as follows. Let us consider 3 cutting segments $x = [AB]$, $y = [CD]$, $z = [EF]$, [2], fig. 2. The segments are oriented in a common way, say, toward the centre of the hexagon $abcdef$. The segments $[AB]$, $[CD]$, $[EF]$ are extended to the closed cycles on T^2, these circles are not inside the first sheet. Cycles are pairwise intersected in a central point of the disk, which is the third sheet of the surface. Also one may consider an alternative extension of the segments inside the first sheet. Say, the segment $[AB]$ (on Fig. 3, this segment is on the first sheet above a half of the segment $[PO]$, which is on the second sheet) is extended by the fold of the first sheet from the right of the point O trough the middle point of the segment $[KL]$ and the point M. One may see that this closed curve on the plane has a single self-intersection point and is on the first sheet. Cycles y and z are defined analogously.

An arbitrary oriented cycle on the surface is decomposed into a sum of x, y; y, z, or z, x–cycles. Say, let us consider a curve l_{y-z} in the homology class $l_y - l_z$. This is a horizontal curve of ∞-shape, which intersects the segments $[CD]$ (with positive sign) and $[EF]$ (with negative sign). Half of the curve l_{y-z} is embedded into the left part of the first sheet (fig. 2), the last part is embedded in the right sheet. On the plane, we get a self-intersection point of l_{y-z} inside the hexagon $abcdef$.

A collection of 3 mirror symmetries of the first sheet of the torus along the cutting segments $[AB]$, $[CD]$, $[EF]$ is well-defined. These symmetries correspond to the reflections R_x, R_y, R_z, see the formula (2). The reflexion R_x keeps the x coordinate and permutes y and z coordinates; analogously, for R_y, R_z. Because the segments are extended to the closed circle l_x, l_y, l_z on T^2, a reflection of the collection is extended to the reflection of the torus. The composition of the two different symmetries is the rotation of angle $\frac{2\pi}{3}$ around the central axis, which is perpendicular to the plane of the projection. This rotation corresponds to the transformation S by the formula (1). In particular, we get: $S(l_x) = l_y$, $S(l_y) = S^2(l_x) = l_z$, $S(l_z) = S^3(l_x) = l_x$. The factorization with respect to the symmetry $S : T^2 \to T^2$ of the Konstantinov torus determines a branching 3-sheeted covering $T^2 \to S^2$ with 3 branching points of the order 3. This points correspond to the intersections of the vertical axis of rotation S with first sheet and with the third sheet (the disk) of the torus. The 3 intersection points of the axis with the second sheet (the cylinder) are cyclically permuted by S.

Let us pass from the projection $\pi \circ \varphi'$ to the immersion φ'. This lift is not canonical and we do the deformation in two steps. The first step is a vertical deformation of the first (and the third) sheet up and the second sheet down with respect to the axis of the projection π. The fold curve is not deformed. On the deformed surface, a self-intersection curve appears, this curve is on fig. 5. The thick black segments correspond to the self-intersection of the first deformed sheet; the thin segments correspond to the self-intersection of the second deformed sheet. The third deformed sheet is an embedded disk, this disk has no intersections with the first and the second sheet. Let us investigate the deformation of the first sheet. On the picture, the thick black segment $[b_6 b_1]$ joins points a and b (but not b and c) from fig. 2. This means that we deform the interior of the right hexagon $a'b'c'd'e'f'$ in the up direction, then the left hexagon $abcdef$. This deformation is chiral. After a small generic deformation of step 2, we get a generic immersion with generic self-intersection as in fig. 6. The self-intersection curve is the black line: it has 3 components.

Let us briefly recall the standard approach for self-intersections and define the self-intersection curve of the immersion φ' formally. Each self-intersection point (except triple points) contains exactly 2 different preimages on the torus T^2. The collection of self-intersection points of φ' in the target space $\mathbb{R}^3$ is the closed curve (with triple self-intersections), this curve is denoted by Δ. Below we will not use the projection $\pi \circ \varphi'$ of the Konstantinov immersion, only the modification φ of the immersion φ', we avoid the collision of notations. The canonical 2-sheeted covering $\pi : \bar{\Delta} \to \Delta$ is well-defined. Formally, using two-point configuration space $T^2 \times T^2$, we get: $\bar{\Delta} \subset T^2 \times T^2 \setminus diag$, where $diag = \{(x,x) | x \in T^2\}$; $\bar{\Delta} = (x,y) \in T^2 \times T^2|_{\varphi(x)=\varphi(y)}$; $\Delta = [x,y], (x,y) \in \bar{\Delta}$. The canonical double covering $\bar{\Delta}$ consists of ordered points with a common image by φ, the curve Δ is the base of the double covering $\pi : \bar{\Delta} \to \Delta$, $\pi(x,y) = [x,y]$.

By an argument concerning the orientations, we get that the canonical covering $\bar{\Delta}$ is disconnected and equipped with canonical orientation. Denote the branches of the canonical covering by $\bar{\Delta} = \bar{\Delta}_a \cup \bar{\Delta}_b$. Take a point $z \in \Delta$ with the preimages, $a \in \bar{\Delta}_a$, $b \in \bar{\Delta}_b$. Let us consider the positive normal vectors $\vec{n}_a$, $\vec{n}_b$ to the immersed oriented torus at points $\varphi(a)$ and $\varphi(b)$. To define the canonical orientation on $\bar{\Delta}$ in the point $a \in \bar{\Delta}$, take the collection of two vectors $\{\vec{n}_a, \vec{n}_b\}$ and define the orientation vector $\vec{e}_a$ along the branch of $\bar{\Delta}_a$ at a, such that the ordered triplet $\{\vec{n}_a, \vec{n}_b, \vec{e}_a\}$ is the positive orientation of $\mathbb{R}^3$. The orientation $\vec{e}_b$ along the branch $\bar{\Delta}_b$ is analogous. The orientation of branches of the covering $\bar{\Delta}$ are opposite to each other on Δ.

A modification of the Konstantinov immersion

The Konstantinov generic immersion on fig. 6 has a self-intersection curve that is not connected. Let us describe a deformation of the first sheet along the vertical axis of the projection, which keeps the projection. As a result, we get the Konstantinov immersion

$$\varphi : T^2 \looparrowright \mathbb{R}^3 \tag{6}$$

with a connected self-intersection curve. By the description above, the hexagon $abcdef$ is below the hexagon $a'b'c'd'e'f'$ with respect to the projection, except three thin domains near the segments $[a,b]$, $[cd]$, $[ef]$, where the corresponding branches of the self-intersection curve Δ are found. Let us consider the triod τ_0 (the graph with 3 boundary vertexes and one triple central vertex and with 3 edges) on the plane of projection with the central triple point at the center of hexagons and the vertexes on the central points of the branches of Δ. The two triods τ and τ' are on the first sheet of the torus on the hexagons $abcdef$ and $a'b'c'd'e'f'$ correspondingly. The triod τ is below the triod τ' and between the two triods with respect to the vertical axis we have no extra sheets of the surface. The vertical down deformation of a thin neighbourhood of τ' to τ is well-defined. By this deformation, the branches of the self-intersection curve Δ are reconnected into a single connected self-intersection curve. We will consider below only the modified Konstantinov immersion and we keep the same notations φ, Δ, etc. By Konstantinov immersion, we will mean the modified Konstantinov immersion.

Lemma 1. *The connected self-intersection curve Δ of the Konstantinov immersion* (6) *satisfies the following additional property: the branches $\bar{\Delta}_a$, $\bar{\Delta}_b$ of the canonical covering, equipped with the canonical orientation, are trivial cycles $[\bar{\Delta}_a], [\bar{\Delta}_b] \in H_1(T^2;\mathbb{Z})$.*

Proof of Lemma 1.

Let us consider the curve Δ of the immersion φ' before the modification $\varphi' \mapsto \varphi$. The curve Δ is decomposed into 3 components: $\Delta(\varphi') = \Delta_1(\varphi') \cup \Delta_2(\varphi') \cup \Delta_3(\varphi')$, see fig. 6. The branches $\bar{\Delta}_{1,b}(\varphi')$, $\bar{\Delta}_{2,b}(\varphi')$, $\bar{\Delta}_{3,b}(\varphi')$ of the canonical covering are inside the second and the third sheets of the surface (the union of the two sheets of the surface along the fold circle is the disk), therefore the oriented curve $\bar{\Delta}_a$ represents the trivial cycle. The branches $\bar{\Delta}_{1,a}$, $\bar{\Delta}_{1,a}$, $\bar{\Delta}_{1,a}$ represent non trivial cycles $x-y$, $y-z=-x$, $z-x=y$ correspondingly. The sum of these 3 cycles is the trivial cycle. The explicit calculation of the homology class of $\bar{\Delta}_a$ can be omitted, because it is well-known that the self-intersection curve represents the trivial oriented cycle for an arbitrary immersed oriented surface. By the modification $\varphi' \mapsto \varphi$ the connected branches $\bar{\Delta}_a$, $\bar{\Delta}_b$ represent the same cycles. This proves that the cycles $[\bar{\Delta}_a(\varphi)]$, $[\bar{\Delta}_b(\varphi)]$ are trivial. $\square$

1.3 Linking numbers

Consider two closed oriented disjoint curves $L_1 = \varphi_1 : S^1 \subset \mathbb{R}^3$, $L_2 = \varphi_2 : S^1 \subset \mathbb{R}^3$. The linking number $k(L_1, L_2) \in \mathbb{Z}$ is the integer algebraic number of intersection points of L_1 with an oriented Seifert surface Σ_2 for L_2. The linking number is symmetric: $k(L_1, L_2) = k(L_2, L_1)$. We will need a standard generalization of linking numbers for curves with intersections.

Assume that the curves $L_1 \subset \mathbb{R}^3$ and $L_2 \subset \mathbb{R}^3$ have a single intersection point $x \in \mathbb{R}^3$. Take the two generic alterations $L_1^- = \varphi_1^- : S^1 \subset \mathbb{R}^3$ and $L_1^+ = \varphi_1^+ : S^1 \subset \mathbb{R}^3$ of the curve L_1, which resolve the singular point x in two different ways. The linking numbers $k(L_1^-, L_2)$, $k(L_1^+, L_2)$ are well-defined. Define $k(L_1, L_2) \in \mathbb{Z}[\frac{1}{2}]$, $\mathbb{Z}[\frac{1}{2}] = \{\dots, -1, -\frac{1}{2}, 0, \frac{1}{2}, \dots\}$, as a half-integer by the formula:

$$k(L_1, L_2) = \frac{1}{2}(k(L_1^-, L_2) + k(L_1^+, L_2)). \tag{7}$$

Assume that the curves $L_1 \subset \mathbb{R}^3$, $L_2 \subset \mathbb{R}^3$ have s common intersection points $\{x_1, \dots, x_s\} \in \mathbb{R}^3$. Take the set $\aleph$ of the 2^s generic alterations of the curve L_1, which resolve the collection of singular points $\{x_1, \dots, x_s\}$ in all possible ways. The collection of 2^s linking numbers, which corresponds to the different resolutions L_1^α, $\alpha \in \aleph$ is well-defined. Define $k(L_1, L_2)$ by the formula:

$$k(L_1, L_2) = \sum_{\alpha \in \aleph} \frac{1}{2^s} k(L_1^\alpha, L_2). \tag{8}$$

Assume we get two curves L_1, L_2 with s intersection points $\{x_1, \dots, x_s\}$, and a projection of the curves onto a plane. Denote by $\{y_1, \dots, y_s\}$ the projections of the singular points, which are the projections of $\{x_1, \dots, x_s\}$, and by $\{z_1, \dots, z_t\}$ the set of self-intersection points of the projection (regular points of the projection), except points in $\{y_1, \dots, y_s\}$. The two collections of algebraic numbers: $o(y_i) = \pm 1$, $i = 1, \dots, s$, $o(z_j) = \pm 1; 0$, $j = 1 \dots t$, of singular points $\{y_1, \dots, y_s\}$ and regular points $\{z_1, \dots, z_t\}$ are well-defined. A sign $o(y_i)$ of a singular point y_i corresponds to the orientation of the ordered 2-bases by the tangent vectors to the projections of the curves L_1, L_2, the first vector of the base is the tangent vector to the projection of L_1. A sign $o(z_j)$ of a regular point z_j of the projection is equal to zero, $o(z_j) = 0$, when the branch L_1 is above the branch L_2 near the point z_j. When L_2 is above, we have $o(z_j) = \pm 1$. We get $o(z_j) = +1$, if the ordered base of the tangent vectors to the branches of the projections is positive with respect to the orientation on the plane.

Proposition 2. *The linking number $k(L_1, L_2) \in \mathbb{Z}[\frac{1}{2}]$ is a half-*

integer. The following formula is satisfied:

$$k(L_1, L_2) = \sum_i \frac{o(y_i)}{2} + \sum_j o(z_j). \tag{9}$$

Proof of Proposition 2.

The left-hand side of the formula (9) contains terms of two types, which gives contribution to y_i and z_j. Assume for simplicity that all algebraic signs $o(y_i)$ are positive. This assumption gives no restriction of generality, because the terms corresponding to a pair of points with different signs cancel. The sum of terms of the first type is given by the formula:

$$\frac{1}{2^s}[1 \cdot C_s^1 + 2C_s^2 + \dots + sC_s^s] = \frac{s2^{s-1}}{2^s}.$$

The sum of terms of the second type is given by the second term in the right-hand side of the formula (9). □

The formula (9) allows to define the linking number $k(L_1, L_2) \in \mathbb{Z}[\frac{1}{2}]$ for two generic curves $L_1, L_2 \looparrowright T^2 \looparrowright \mathbb{R}^3$ on immersed oriented torus. We assume that the set of intersections of L_1 with L_2 on T^2 contains only finite number of isolated points $\{x_1, \dots, x_s\}$. Then we may assume that all the points $\{x_1, \dots, x_s\}$ are inside the disk $D^2 \subset T^2$. Take the projection of the immersed torus on the plane and assume that the image of D^2 is a regular disk on the plane (we assume that the orientation of the projection of the disk on the plane and the orientation of the disk on the torus coincide). We may apply Proposition 2 to calculate $k(L_1, L_2)$. The sign $o(x_i)$, $i = 1, \dots, s$ of a singular point equals to the algebraic sign of intersection of the curves L_1 and L_2 on T^2.

This sign is changed, when the order of the curves L_1, L_2 is changed. The linking number does not depend on the order. This is not clear from (9), because the second terms in the right-hand side of (9) is not symmetric, and is not skew-symmetric with respect to the permutations of the order.

In the case we change the orientation on T^2, the orientation of the projection of the disk is opposite to the orientation of the plane. In this case, contributions of intersection points in the formula (9) remain unchanged, because the signs $o(x_i)$ is changed to the opposite twice: the intersection index of the cycles L_1 and L_2 on T^2 at the point x_i is changed, and the contribution of the sign of the intersection of the curves on the torus will be taken as opposite.

If we take a composition of the immersion $\varphi : T^2 \looparrowright \mathbb{R}^3$ with the reflection $\mathbb{R}^3 \to \mathbb{R}^3$ with the axis parallel to the plane of the projection, the linking number (9) is change to the opposite, because all the terms in the right-hand side of the formula are changed. $\square$

2. Perfect immersions

Let us define a q-perfect $q \in \{0, +1, \dots\}$ (generic) immersion $\varphi : T^2 \looparrowright \mathbb{R}^3$ of the torus in the Euclidean space. Take the self-intersection curve $\Delta(\varphi)$ of the immersion φ. Decompose this curve into connected components: $\Delta = \cup_{i=1}^{s} \Delta_i$. A component Δ_i admits the canonical covering $\pi_i : \bar{\Delta}_i \to \Delta_i$. The collection of the curves $\{\bar{\Delta}_{1,a}, \bar{\Delta}_{1,b}, \dots \bar{\Delta}_{i,a}, \bar{\Delta}_{i,b}\}$ determines the collection of integer homology classes $\{[\bar{\Delta}_{1,a}], [\bar{\Delta}_{1,b}], \dots [\bar{\Delta}_{s,a}], [\bar{\Delta}_{s,b}] \in H_1(T^2; \mathbb{Z})\}$. Each integer class $x \in H_1(T^2; \mathbb{Z})$ is a pair of integers (n_x, m_x). Therefore, the collection of $4s$ integers: $\{n_{1,a}, m_{1,a}, n_{1,b}, m_{1,b}, \dots, n_{s,a}, m_{s,a}, n_{s,b}, m_{s,b}\}$.

Definition 3. *1. We shall say that an immersion $\varphi : T^2 \looparrowright \mathbb{R}^3$ is q-perfect, if the greatest common divisor of the integers: $\{n_{1,a}, m_{1,a}, n_{1,b}, m_{1,b}, \dots, n_{s,a}, m_{s,a}, n_{s,b}, m_{s,b}\}$ is divisible by q. In the case all the cycles $[\bar{L}_a]$, $[\bar{L}_b]$ are trivial, we say that the immersion is also ∞-perfect.*

2. We call two immersions $\varphi_1 : T^2 \looparrowright \mathbb{R}^3$, $\varphi_2 : T^2 \looparrowright \mathbb{R}^3$ are q-perfect regular homotopic, if there exists a regular homotopy φ_t, $t \in [0, 1]$, which coincides with the two given immersions for $t = 0, t = 1$ with finite number reconnection points of self-intersection curves, such that an arbitrary immersion φ_t of the regular homotopy is q-perfect.

Example 4.

1. The Konstantinov immersion (6) *is ∞-perfect. This is a corollary of the Lemma 1.*

2. Take the standard embedded torus $i : T^2 \subset \mathbb{R}^3$ and the 4-sheeted covering $p : T^2 \to T^2$, which is double covering along the parallel and the meridian. Let us consider the immersion $i \circ p : T^2 \to T^2$ and deform it into a regular position. We get a 1-perfect immersion (i.e. an immersion with no additional properties concerning its self-intersection), which is regular homotopic to the Konstantinov immersion.

Take a q-perfect immersion $\varphi : T^2 \looparrowright \mathbb{R}^3$. Assume we have cycles $\mathbf{a}, \mathbf{b} \in H_1(T^2;\mathbb{Z})$ and two closed curves $l_{\mathbf{a}} = f_{\mathbf{a}} : S^1 \looparrowright T^2$, $l_{\mathbf{b}} = f_{\mathbf{b}} : S^1 \looparrowright T^2$ in a general position. Take the corresponding curves $L_{\mathbf{a}} = \varphi \circ f_{\mathbf{a}} : S^1 \looparrowright \mathbb{R}^3$, $L_{\mathbf{b}} = \varphi \circ f_{\mathbf{b}} : S^1 \looparrowright \mathbb{R}^3$ with a finite number of intersection points $\{x_1, \dots, x_s\}$, which correspond to intersection points of curves $l_{\mathbf{a}}$, $l_{\mathbf{b}}$ on T^2. A half-integer linking number $k(L_{\mathbf{a}}, L_{\mathbf{b}}) \in \mathbb{Z}[\frac{1}{2}]$ is well-defined. This number depends on the curves $l_{\mathbf{a}}$, $l_{\mathbf{b}}$ in the homology classes $[l_{\mathbf{a}}] \in H_1(T^2;\mathbb{Z})$, $[l_{\mathbf{b}}] \in H_1(T^2;\mathbb{Z})$. Let us define an integer $\sharp(l_{\mathbf{a}}, l_{\mathbf{b}}) \pmod q$, such that the sum

$$LK(L_{\mathbf{a}}, L_{\mathbf{b}}; \varphi) = k(L_{\mathbf{a}}, L_{\mathbf{b}}) + \sharp(l_{\mathbf{a}}, l_{\mathbf{b}}) \in \mathbb{Z}_q[\frac{1}{2}] \qquad (10)$$

depends only on the homology classes $[l_{\mathbf{a}}]$, $[l_{\mathbf{b}}]$ and of φ. Above by $\mathbb{Z}_q[\frac{1}{2}]$ the half-integers modulo q is defined. This formula is well-defined for in the case, when $L_{\mathbf{a}}$, $L_{\mathbf{b}}$ are non-connected, are represented by a finite collection of oriented closed circles on the torus. In this case we may apply the formula (10) to each pairs of connected components of the curves $L_{\mathbf{a}}$, $L_{\mathbf{b}}$.

2.1. Definition of a disorder number $\sharp(l_{\mathbf{a}}, l_{\mathbf{b}}) \pmod q$ of two generic curves $l_{\mathbf{a}}, l_{\mathbf{b}}$ on an immersed q-perfect torus

The curves $L_{\mathbf{a}}$, $L_{\mathbf{b}}$ in $\mathbb{R}^3$ could get an intersection point, when $l_{\mathbf{a}}$, $l_{\mathbf{b}}$ deforms in its homotopy classes on T^2. This intersection point corresponds to a common point on the self-intersection curve $\Delta(\varphi)$, one of the curve, say $l_{\mathbf{a}}$, is deformed along one of the two different branches of $\bar{\Delta}$, say along the branch Δ_a. Assume that the projection $\pi(\bar{\Delta}_a)$ on Δ of the intersection point $l_{\mathbf{a}}$ with Δ_a coincides to the projection on Δ of an intersection point of $l_{\mathbf{b}}$ with $\bar{\Delta}_b$. By this intersection the linking number $k(L_{\mathbf{a}}, L_{\mathbf{b}})$ admits a jumps $\sigma(z) = \pm 1$. The sign depends only on the sign of the intersection point of $l_{\mathbf{a}}$ with $\bar{\Delta}_a$, on the sign of the intersection point of $l_{\mathbf{b}}$ with $\Delta_{\mathbf{b}}$ and on the orientation of $\bar{\Delta}_a$, or, equivalently, on the orientation of $\bar{\Delta}_b$ near the intersection point. We say that the intersection of $L_{\mathbf{a}}$ and $L_{\mathbf{b}}$ with the sign ± 1, is compensated by the disorder number. Let us give a formal definition of a disorder number $\sharp(l_{\mathbf{a}}, l_{\mathbf{b}})$.

Let us consider an arbitrary branch Δ_i of the curve Δ. Let us consider the canonical covering $\bar{\Delta}_{i,a} \cup \bar{\Delta}_{i,b} \to \Delta_i$. Let us denote the set of intersection points of $l_{\mathbf{a}}$ with $\Delta_{i,a}$ by $\{a_1, \dots, a_s\}$, the set of intersection points of $l_{\mathbf{b}}$ with $\bar{\Delta}_{\mathbf{b}}$ by $\{b_1, \dots, b_t\}$, the set of intersection points of $l_{\mathbf{a}}$ with $\bar{\Delta}_{\mathbf{b}}$ by $\{\bar{a}_1, \dots, \bar{a}_p\}$, the set of intersection points of $l_{\mathbf{b}}$ with $\bar{\Delta}_{\mathbf{a}}$ by $\{\bar{b}_1, \dots, \bar{b}_r\}$. Each considered point is equipped with a sign ± 1, because this is an intersection point of the two oriented curve on T^2. The sum of the signs of points of the each set is divided by q, because the cycles $[\bar{\Delta}_a]$ $[\bar{\Delta}_b]$ in $H_1(T^2; \mathbb{Z}_q)$ are trivial.

We may consider the collection $\{\bar{b}_1, \dots, \bar{b}_r\}$ as the collection of points on $\bar{\Delta}_{\mathbf{b}}$, because $\bar{\Delta}_{\mathbf{a}}$ and $\bar{\Delta}_{\mathbf{b}}$ are naturally identified as the branches of the covering over the common base circle. The same way we may consider the collection $\{\bar{a}_1, \dots, \bar{a}_p\}$ as the collection of points on $\bar{\Delta}_{\mathbf{b}}$. Assume that, on the circle $\bar{\Delta}_a$, the collections $\{a_1, \dots, a_s\}$, $\{\bar{b}_1, \dots, \bar{b}_r\}$ are inside two disjoint segments. Also assume that on the circle $\bar{\Delta}_b$, the collections $\{b_1, \dots, b_t\}$, $\{\bar{a}_1, \dots, \bar{a}_p\}$ are inside two disjoint segments. We shall say that the intersection points are in ordered position on Δ_i. In this case, we shall tell that the contribution to the disorder number $\sharp(l_{\mathbf{a}}, l_{\mathbf{b}}) = 0 \pmod q$ from the component Δ_i is trivial. In the case of an arbitrary position of $\{a_1, \dots, a_s\}$, $\{\bar{b}_1, \dots, \bar{b}_r\}$ on $\bar{\Delta}_a$, of $\{b_1, \dots, b_t\}$, $\{\bar{a}_1, \dots, \bar{a}_p\}$ on $\bar{\Delta}_b$, the algebraic numbers $\sharp_{a,i}$, $\sharp_{b,i}$ of permutations of the collections of points to the ordered positions along Δ_i are well defined. The sign of a permutation is positive, if by this permutation the linking number $k(L_{\mathbf{a}}, L_{\mathbf{b}})$ changes by -1, and the sign of a permutation is negative in the opposite case.

Definition 5. *Let us define* $\sharp(l_{\mathbf{a}}, l_{\mathbf{b}}) = \sum_i \sharp_{a,i} + \sharp_{b,i}$.

The integer (10) is well defined modulo q, because the term $\sharp(l_{\mathbf{a}}, l_{\mathbf{b}})$ in the left-hand side of the formula is well-defined modulo q. Moreover, a q-perfect regular homotopy of φ preserves the integer (10).

The following is a corollary of our results on Kempe dynamic.

Corollary 6.

The Konstantinov immersion (6) *is not* 2*-perfect regular homotopic to a mirror image by a reflection with respect to a plane. More detailed:* $\varphi : T^2 \looparrowright \mathbb{R}^3$, $\Sigma \circ \varphi : T^2 \looparrowright \mathbb{R}^3$, *where* $\Sigma : \mathbb{R}^3 \to \mathbb{R}^3$ *is the reflection with respect to the plane of the projection of the Konstantinov immersed torus (this reflection keeps the projection of the torus and inverse the orientation of the vertical axis), are* ∞*-perfect immersion, which are regular homotopic, but not* 2*-perfect regular homotopic.*

Corollary 7. *The Konstantinov immersion is not* 4*-perfect regular homotopic to a inside-out image. More detailed:* $\varphi : T^2 \looparrowright \mathbb{R}^3$ *and* $\varphi \circ R : T^2 \looparrowright \mathbb{R}^3$, *where* $R = R_z : T^2 \to T^2$ *is the diffeomorphism* (2)*, which reverses the orientation on the torus, are* ∞*-perfect immersions, which are regular homotopic, but not* 4*-perfect regular homotopic.*

3. A visualization of stable invariants

The idea of the visualization of the stable invariant I_2 is following. We assume that the Λ-lattice is realized as a Λ-lattice on the immersed Konstantinov torus $\varphi : T^2 \looparrowright \mathbb{R}^3$. Then we express the invariants $I_2 = \pm 1$ as twice the linking number, $2LK(L_{\mathbf{a}}, L_{\mathbf{b}}) \pmod 4$ of two flux loops, representing cycles **a** and **b** of a Kempe sector with $\chi(\mathbf{a}, \mathbf{b}) = 1 \pmod 2$. In this construction, the following property is required: an oriented loop on T^2, which represents an arbitrary basic cycles in $H_1(T^2; \mathbb{Z})$ (by a basic cycle we means an arbitrary cycle, which is not in the kernel of the modulo 2 reduction homomorphism $H_1(T^2; \mathbb{Z}) \to H_1(T^2; \mathbb{Z}_2)$) is self-linked in $\mathbb{R}^3$ with an odd coefficient. Therefore, because a disorder number of a curve with its parallel copy by a small alteration is even, a pair of cycles in a common basic homology class is linked with the odd coefficient. A reflection with respect to a plane changes the double of the linking number $2LK(L_{\mathbf{a}}, L_{\mathbf{b}}) = +1$ to the antipodal number -1. This is not possible by a 2-perfect regular homotopy, because such a homotopy keeps twice the linking number modulo 4.

3.1. A visualization of χ

Let us start with χ. We take oriented cycles $\mathbf{a}, \mathbf{b}$ on T^2 in a general position. The number of intersection points of the two cycles modulo 2 corresponds to $\chi(\mathbf{a}, \mathbf{b}, \mathbf{c})$. This is proved in [1], fig. 7. To investigate $I_2(\mathbf{a}, \mathbf{b}, \mathbf{c})$, we have to assume that $\chi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$. This implies that the assumption (4) is satisfied, up to the permutation of cycles $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

3.2. A visualization of I_2

Let us introduce a reflection $\rho : T^2 \to T^2$, $\rho^2 = Id : T^2 \to T^2$ which keeps the x-coordinate and reverses the y-coordinate: $\rho(x) = x$, $\rho(y) = -y$. Note, the involution R_x, given by (2), satisfies the condition $\rho(x) = x$, but not the condition $\rho(y) = -y$, because $R_x(y) = z$. This symmetry ρ is not an isometric transformation of the torus, this transformation preserves vertexes of the Λ-lattice on T^2, but edges of the Λ-lattice are not ρ-invariant. One may assume that transformation ρ is a free involution, the quotient T^2/ρ is a Klein bottle.

Take the cycles $\mathbf{a}, \mathbf{b} \in H_1(T^2; \mathbb{Z})$, which are defined by the corresponding oriented, probably, disconnected, curves $l_{\mathbf{a}}, l_{\mathbf{b}}$. Consider the mirror image $\rho(l_{\mathbf{b}})$ and assume that $l_{\mathbf{a}}$ and $\rho(l_{\mathbf{b}})$ are in a general position. Consider the curves $L_{\mathbf{a}} = \varphi(l_{\mathbf{a}})$, $R_z(L_{\mathbf{b}}) = \varphi(\rho(l_{\mathbf{b}})) \looparrowright \mathbb{R}^3$ as the images of cycles $l_{\mathbf{a}}, \rho(l_{\mathbf{b}})$ by φ, these curves have generic intersections, because $l_{\mathbf{a}}$ and $\rho(l_{\mathbf{b}})$ are generic curves on the torus.

The integer $2LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi) \pmod 4$ is well-defined by the equation (10). Twice the disorder number $2\sharp(l_{\mathbf{a}}, l_{\mathbf{b}})$ gives only even contributions.

Let us prove that the integer $2LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi) \pmod 4$ coincides with the integer given by equation (3). Both integers are bilinear with respect to cycles $\mathbf{a}$, $\mathbf{b}$. In the case $[l_{\mathbf{a}}] = [l_{\mathbf{b}}] = l_x \pmod 2$, or $[l_{\mathbf{a}}] = [l_{\mathbf{b}}] = l_y \pmod 2$ the integer (3) equals to 2. The integer

$$2LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi) \pmod 4 \tag{11}$$

coincides to (3), because of the following property of the Konstantinov immersion: an arbitrary generic cycle (a cycle is generic if it is not a boundary modulo 2) is self-linked with odd coefficient.

We may assume that the condition (4) is satisfied modulo 4, this gives no restriction. Let us prove this. Both numbers, $LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi)$ $\pmod 4$ and (3) are not changed by a permutation of cycles. For the number $LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi)$, this follows from the fact that, with the assumption (4), we get $LK(L_{\mathbf{b}}, \rho(L_{\mathbf{a}}); \varphi) \pmod 4 = LK(-L_{\mathbf{a}}, \rho(-L_b); \varphi)$ $\pmod 4$, where $-L$ is L with the opposite orientation.

The number $LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi)$ is preserved by a cyclic permutation of $\mathbf{a}, \mathbf{b}, \mathbf{c}$, because the linking number of two loops on the Konstantinov torus is invariant with respect to the rotation S of the torus. The number (3) is preserved by a cyclic permutation, because this number is represented in a symmetric form (42) [1].

With the condition (4) modulo 4 the numbers $2LK(L_{\mathbf{a}}, \rho(L_{\mathbf{b}}); \varphi)$ $\pmod 4$ and (3) are odd. The first number is skew-invariant with respect to a reflection in $\mathbb{R}^3$, the second is invariant. We may take one of the two mirror copies of the Konstantinov immersion, for which the two numbers coincide.

This gives a visualization of the invariant I_2 by linking numbers of the flux loops in $\mathbb{R}^3$.

3.3. A visualization of I^+ in the sector $I_2 = +1$

Take the cycles $\mathbf{a}, \mathbf{b}, \mathbf{c} \in H_1(T^2; \mathbb{Z})$, which are defined by the corresponding curves $l_{\mathbf{a}}, l_{\mathbf{b}}, l_{\mathbf{c}}$. We assume that we are in the sector $I_2(\mathbf{a}, \mathbf{b}, \mathbf{c}) = +1$, and the notations correspond to the restriction (4). First we rewrite this formula as follows:

$$I^+(\{\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}\}) \pmod 4 = \frac{1}{2}(\tilde{a}_y \tilde{b}_x + 2\tilde{b}_x - (l_{\mathbf{a}} \cap l_x + 3l_{\mathbf{b}} \cap l_y + l_{\mathbf{c}} \cap l_z)), \tag{12}$$

where $l_{\mathbf{a}} \cap l_x$ is an intersection number of the oriented cycles on the oriented torus, equipped with a prescribed orientation O. In the present form, the formula is satisfied without the condition (4), because an odd permutation of the coordinates (x, y, z) changes the orientation O and the signs of the product. The terms $\frac{1}{2}(\tilde{a}_y \tilde{b}_x + 2\tilde{b}_x)$ is even and is invariant to its antipodal. Consider the mirror image $\rho(l_{\mathbf{b}})$ and assume that $l_{\mathbf{a}}$ with $l_{\mathbf{b}}$ (and with $\rho(l_{\mathbf{b}})$) is in a general position. Consider the curves $L_{\mathbf{a}}, L_{\mathbf{b}}, \rho L_{\mathbf{b}} \looparrowright \mathbb{R}^3$ as the images of the cycles $l_{\mathbf{a}}, l_{\mathbf{b}},\ \rho(l_{\mathbf{b}})$

correspondingly.

$$I^+(\{\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}\}) \pmod 4 =$$
$$\frac{1}{2}(LK(L_{\mathbf{a}}, L_{\mathbf{b}}) + LK(L_{\mathbf{a}}, \rho L_{\mathbf{b}}) - (l_{\mathbf{a}} \cap l_y a + 3l_{\mathbf{b}} \cap l_x + l_{\mathbf{c}} \cap l_z)). \tag{13}$$

The formulas (12) and (13) are equivalent, but the second formula is possible when the linking number LK of cycles is well-defined modulo 4.

Proof of Corollary 6.

Let us compare the invariant I_2 for Kempe dynamic on the Konstantinov torus and on its mirror image by Σ. The topological interpretation of the invariant I_2 proves that the invariants for corresponding sectors have to be opposite (the linking number is skew-symmetric with respect to the mirror image) and coincide simultaneously. This is not possible, if we assume that the Konstantivov torus is 2-perfect regular homotopic to its mirror copy. $\square$

Proof of Corollary 7.

Let us start by the following observation, which is interesting by itself and will be used in the proof. The equation (12) is a way to determine an orientation O on the torus. In the case the condition (4) is defined up to a cyclic permutation of the coordinates x, y, z, (this cyclic permutation corresponds to the orientation-preserved rotation (1)). In the case the condition (4) is broken with an odd permutation (this odd permutation corresponds to orientation-reversed rotations (2)), a cyclic order of the coordinates x, y, z is changed to the antipodal and the equation (12) is changed into the antipodal.

Let us compare the invariant I^+ by the formula (13) for cycles $\mathbf{a}, \mathbf{b}, \mathbf{c}$ on the Konstantinov φ torus and on its image $\varphi \circ R_x$ by the involution $R_x : T^2 \to T^2$. The first term $LK(L_{\mathbf{a}}, L_{\mathbf{b}})+LK(L_{\mathbf{a}}, \rho L_{\mathbf{b}})$ is invariant for the transformation $x \leftrightarrow y$. The second term $l_{\mathbf{a}} \cap l_y a + 3l_{\mathbf{b}} \cap l_x + l_{\mathbf{c}} \cap l_z$ is skew-invariant for the transformation $x \leftrightarrow y$, because this transformation corresponds to R_z and changes the orientation O on the torus. This proves that (12) and (13) are changed by the regular homotopy from φ to $\varphi \circ R_z$. This proves that such a regular homotopy is not 4-perfect. $\square$

4. Stable homotopy groups of spheres and dynamics with internal symmetries, ergodicity and unstable invariants of dynamics: discussion

By the Pontryagin-Thom construction the Konstantinov immersion represents the generator of the second stable homotopy group of spheres, Π_2. This allows for the visualization of I_2. First, this is the generator and all 3 basic cycles l_x, l_y, l_z are self-linked with odd coefficient, with this property, we can estimate (11). Second, the Konstantinov immersion is ∞-perfect, in particular, is 2-perfect. The standard model of the generator of Π_2 is given by the 4-sheeted covering over the standard torus in $\mathbb{R}^3$, it is not 2-perfect, see Example 4. There are many other examples of G-immersed surfaces (up to regular G-cobordism) with a discrete normal bundle G-structure, which are motivated by homotopy theory. The example of the hyperquaternionic Klein bottle [3] represents a generator of order 16, which describes the stable homotopy group $\Pi_7(\mathbb{RP}^7)$. Another generator of order 2 of this group is represented by a Boy immersion (with internal symmetry).

Ergodicity could help to classify the unstable invariants. We reformulate the periodic problem from the plane to the Lobachevskii plane. This gives more tools: the asymptotic properties of the flux loops on the universal (ramified) hyperbolic plane over the Euclidean plane could be investigated using the ergodic theorem. This gives some information about the unstable invariants on the torus.

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P. M. Akhmet'ev
National Research University "Higher School of Economics", Moscow, Moscow, Russia; IZMIRAN, Troitsk, Moscow region, Russia;
pmakhmet@mail.ru
Olivier Cépas
Université Grenoble Alpes, CNRS, Grenoble INP, Institut Néel, 38000 Grenoble, France.