Integer Quantum Hall Effect: a bulk perspective

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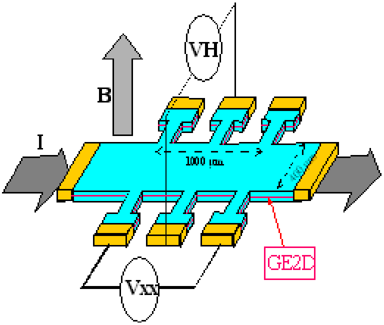
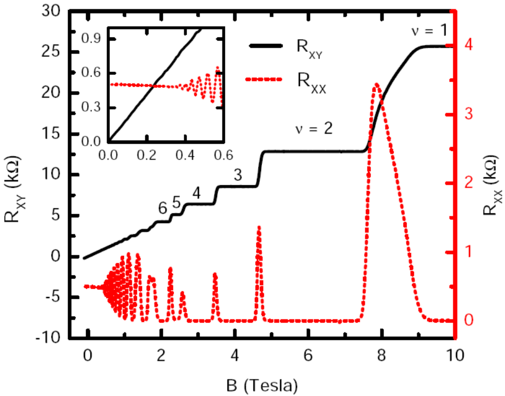
Motivation

What's so special about IQHE?

- High precision quantification of the Hall conductance
- Disorder plays a central and positive role





Why study IQHE now?

Experiments:

- New effects: microwave induced zero-resistance states
- New probes: local sensing techniques
- New systems: graphene, 2d edge states?

Electron interactions play an important part in IQHE also!

Theory:

- Many fundamental aspects well understood
- But: how do we calculate stuff!?

Summary

- Motivation
- Basic elements on IQHE
  - Semiclassical picture
  - Landauer-Büttiker-Halperin edge state picture
  - Landau levels and wavefunctions
- The high field expansion
  - Vortex states and Green's functions
  - Local equilibrium properties
  - Bulk transport equations
- Outlook: conclusion and perspectives

Basics of IQHE:

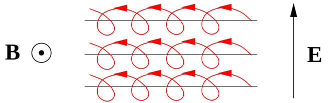
Semiclassical picture

Classical motion in high perpendicular magnetic field

Stable trajectories: $\mathbf{F} = -e\mathbf{v} \times \mathbf{B} - e\mathbf{E} = \mathbf{0}$

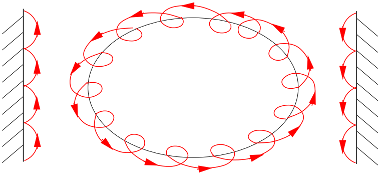
$\longrightarrow$ slow drift velocity: $\mathbf{v}_d = \frac{1}{B}\mathbf{E} \times \hat{\mathbf{z}} = -\frac{1}{B}\nabla V \times \hat{\mathbf{z}}$

Transport: drift dominates over fast cyclotron motion



Disordered bulk: localization on closed equipotential lines

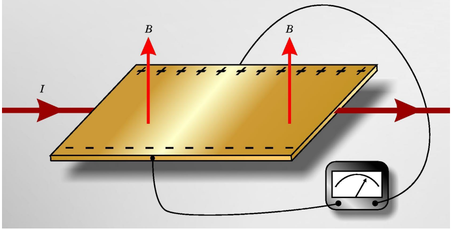
Sharp edges: delocalized skipping orbits



Classical Hall effect

Local current density: $\mathbf{j}(\mathbf{r}) = -en_e(\mathbf{r})\mathbf{v}_d = -\frac{e}{B} n_e \mathbf{E} \times \hat{\mathbf{z}} = -\sigma_{xy} \mathbf{E}$

$\longrightarrow$ local Hall conductivity: $\sigma_{xy}(\mathbf{r}) = \frac{e}{B} n_e(\mathbf{r})$



For a homogeneous sample: $I = G_{xy} V_{xy}$

where $G_{xy} = \frac{e}{B} n_e$ is the Hall conductance

$\longrightarrow$ G_{xy} gives information on the carriers charge and density

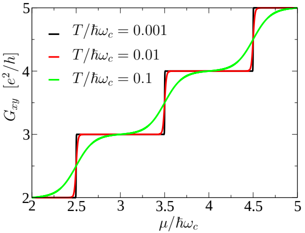
Quantum effects on Hall transport

Back to conductance: $G_{xy} = \frac{e}{B} n_e = \frac{e^2}{h} \frac{h}{eB} n_e = \frac{e^2}{h} \nu$
where $\nu = \frac{h}{eB} n_e = 2\pi l_B^2 n_e$ is a dimensionless density (filling factor)
This defines also a magnetic length: $l_B = \sqrt{\hbar/eB} = 8\text{nm}$ at 10T

Landau level quantization: $\epsilon_m = \hbar\omega_c(m + \frac{1}{2})$
with the cyclotron energy: $\hbar\omega_c = \frac{\hbar eB}{m^\star} = 20$ meV at 10T

Semiclassical picture of IQHE: successive filling of Landau level with integer $\nu = \sum_m n_F(\epsilon_m - \mu)$ leads to successive G_{xy} plateaus:

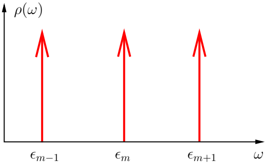
$$G_{xy} = \sum_m n_F(\epsilon_m - \mu)$$

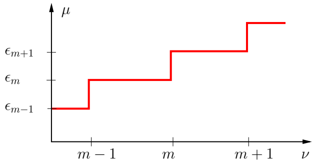


Homogeneous system

As a function of density:

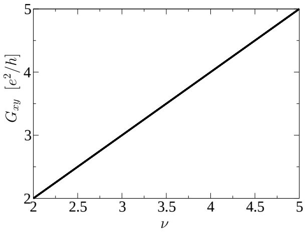
- sharp Landau levels
- μ sticks to ϵ_m and jumps between LL





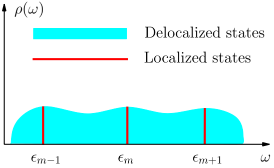
Conductance:

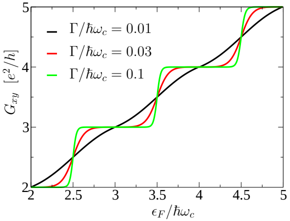
No plateaus!

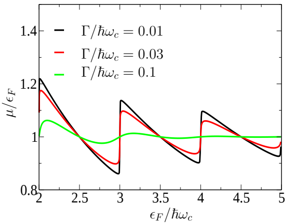


With disorder

<u>With disorder:</u> μ is pinned by bulk localized states



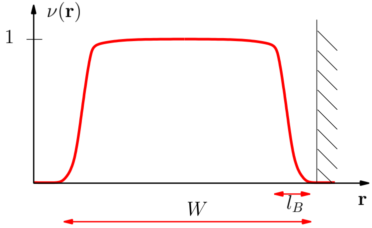




Disorder is essential to plateaus formation!

One more caveat...

Hall bar: quantization of σ_{xy} does not guarantee G_{xy} quantized



- width W ∼1mm
- non-homogeneous region near edge of width l_B ∼8nm

Deviation to quantization: of the order $l_B/W \sim 10^{-5}$

$\longrightarrow$ in contradiction with experiment: $\delta G_{xy}/G_{xy} < 10^{-9}$

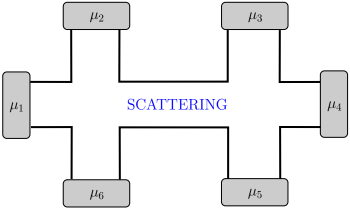
More subtle transport theory is needed!

Basics of IQHE:

Edge state picture

Landauer-Büttiker scattering theory

- Include contacts at fixed chemical potentials μ_i



- Only states near ϵ_F contribute to the current I_i in each contact (diffusion mechanism)
- Büttiker formula:

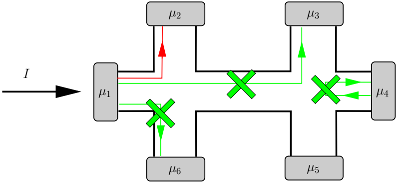
$$I_i = \frac{e}{h}\Big[(\nu_i - R_i)\mu_i - \sum_{j \neq i} T_{ij}\mu_j\Big]$$

where R_i = reflection amplitude in lead i
and T_{ij} = transmission amplitude from lead j to lead i

Edge-state picture

IQHE regime:

- No backscattering of edge states (skipping orbits): $R_i = 0$
- Electrons are transferred from one contact to the next: only $T_{i,i+1} = \nu_i$ non zero

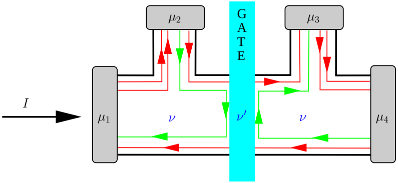


Advantages of this formulation:

- The precise spatial dependence of the potential drop $\Phi(\mathbf{r})$ in the Hall bar is not needed
- Deviations to quantization and dissipative features are related to additional scattering mechanisms

Scattering: a simple example

4 terminals with single "scatterer":



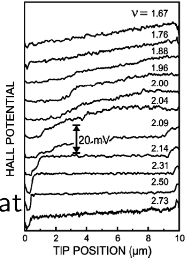
Transmissions:

$T_{2\leftarrow 1} = \nu,\ T_{3\leftarrow 2} = \nu',\ T_{1\leftarrow 2} = \nu - \nu'$, etc...

Resistances:

- $R_{14} = \frac{h}{e^2 \nu'}$: two-point resistance
- $R_{34} = \frac{h}{e^2 \nu}$: Hall resistance
- $R_{23} = \frac{h}{e^2}(\frac{1}{\nu'} - \frac{1}{\nu})$: four-point resistance

Limitations:



Limitations of Büttiker approach to IQHE:

- No information on the local variations of $\Phi(\mathbf{r})$
- Frequency dependent transport difficult to treat
- Edges can leak in the bulk at large bias
- Taking into account disorder: scattering problem very hard
- Calculation of local observables (electronic density, diamagnetic currents) not very practical: $\rho(\mathbf{r}) \sim \mathrm{Tr}\left[\hat{S}^\dagger \frac{\delta \hat{S}}{\delta V(\mathbf{r})}\right]$
- Challenge to incorporate electron interactions

An alternative approach from the bulk is needed!

Basics of IQHE:

Landau levels and wavefunctions

Schrödinger equation in a magnetic field

Free Hamiltonian: no disorder, no interactions

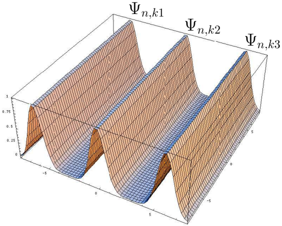
$$H_0 = \frac{1}{2m^\star}\left(-i\hbar\nabla_{\mathbf{r}} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 \quad \text{with} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Landau states:

$$E_{n,k} = \hbar\omega_c(n + \tfrac{1}{2})$$

$$\Psi_{n,k}(x,y) = e^{iky}\exp\left[-\frac{(x-kl_B^2)^2}{2l_B^2}\right] H_n\left(\frac{x-kl_B^2}{l_B}\right)$$

- Translationally invariant along y
- Localized on a scale l_B along x



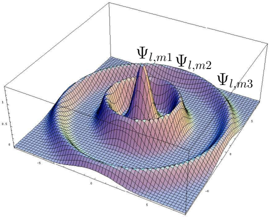
Another solution

Circular states:

$$E_{m,l} = \hbar\omega_c(l + \frac{m+|m|+1}{2}) = \hbar\omega_c(n + \frac{1}{2})$$

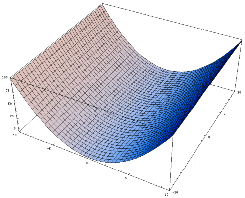
$$\Psi_{l,m}(r,\theta) = e^{im\theta} r^m \exp\left[\frac{-r^2}{4l_B^2}\right] L_l^m\left(\frac{r^2}{2l_B^2}\right)$$

- Rotationally invariant around the origin
- Localized on a scale l_B along r



The absence of an external potential leads to a huge degeneracy!

Confinement



1D Parabolic potential:

$$H = H_0 + V(x) = H_0 + \frac{1}{2} m^\star \omega_0^2 x^2$$

Modified Landau states:

$$E_{n,k} = \hbar\Omega\left(n + \frac{1}{2}\right) + V(kL^2)$$

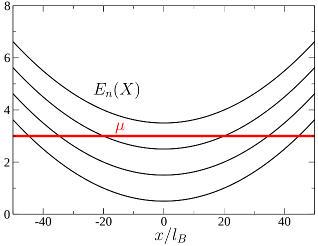
$$\Psi_{nk}(\mathbf{r}) = e^{-iky} \exp\left[-\frac{\left(x - \frac{\omega_c}{\Omega} kL^2\right)^2}{2L^2}\right] H_n\left(\frac{x - \frac{\omega_c}{\Omega} kL^2}{L}\right)$$

where $\Omega = \sqrt{\omega_c^2 + \omega_0^2} \simeq \omega_c$ and $L = \sqrt{\hbar / m^\star \Omega} \simeq l_B$

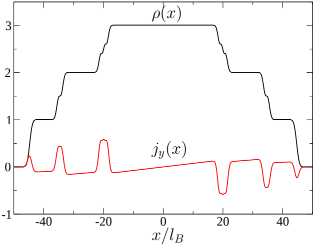
- Degeneracy is fully lifted by $V(x)$
- Wavefunction localize around equipotential lines: $X = kl_B^2$
- Drift velocity: $v_y(X) = \frac{1}{\hbar} \frac{dEn,k}{dk}$

Local equilibrium properties

Three filled LL:



Density and current:

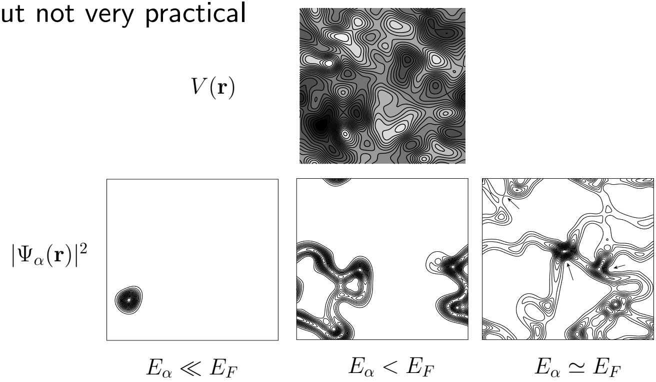


Current originate from drift and density gradient

With disordered potential

<u>Numerical solution:</u>

- Confirms the intuition
- But not very practical



Is there an analytical solution at high field?

The high field expansion:

Vortex states and Green's functions

What is the small parameter?

At large magnetic field:

- Magnetic length: l_B =8nm at 10T
- Correlation length of the disordered potential: $\xi \gtrsim$ 100nm in heterostructures

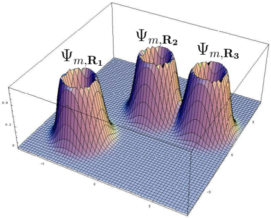
The random potential is smooth on the scale l_B!

Mathematically:

- l_B/ξ is the small parameter
- $V(\mathbf{r})$ can be written in a gradient expansion where $V(\mathbf{r}) \gg l_B|\nabla V(\mathbf{r})| \gg l_B^2|\Delta V(\mathbf{r})| \gg \dots$

What are the correct quantum states?

We need: states that can adapt to an arbitrary shape of $V(\mathbf{r})$, with no preferred symmetry



Vortex states: $\Psi_{m,\mathbf{R}}(\mathbf{r}) = \langle \mathbf{r} | \mathbf{R}, m \rangle$

$$E_{m,\mathbf{R}} = \hbar\omega_c(m + \frac{1}{2})$$

$$\Psi_{m,\mathbf{R}}(\mathbf{r}) = |\mathbf{r} - \mathbf{R}|^m \, e^{im\arg(\mathbf{r}-\mathbf{R})} \exp\left[-\frac{(\mathbf{r}-\mathbf{R})^2 - 2i\hat{\mathbf{z}}\cdot(\mathbf{r}\times\mathbf{R})}{4l_B^2} \right]$$

Remark: this is an overcomplete, coherent state basis

$$\langle \mathbf{R_1}, m_1 | \mathbf{R_2}, m_2 \rangle = \delta_{m_1,m_2} \exp\left[-\frac{(\mathbf{R}_1 - \mathbf{R}_2)^2 - 2i\hat{\mathbf{z}}\cdot(\mathbf{R}_1 \times \mathbf{R}_2)}{4l_B^2} \right]$$

Vortex Green's functions

How to proceed:

- Define $G_{\mathbf{R}_1,m_1;\mathbf{R}_2,m_2}(\omega) = \langle \mathbf{R_1}, m_1|(\omega - \hat{H}_0 - \hat{V})^{-1}|\mathbf{R_2}, m_2\rangle$
- Expand in powers of l_B and gradients of $V(\mathbf{r})$:
  $G = \langle \mathbf{R_1}, m_1|\mathbf{R_2}, m_2\rangle \sum_n (l_B/\sqrt{2})^n g^{(n)}$

Lowest order result: semiclassical limit at $l_B \to 0$!

$$g^{(0)}_{m_1;m_2}(\mathbf{R}) = \frac{\delta_{m1,m2}}{\omega - \epsilon_{m1} - V(\mathbf{R})}$$

To all orders: closed recursion

$$g^{(n)}_{m_1;m_2}(\mathbf{R}) = g^{(0)}_{m_1;m_1}(\mathbf{R}) \sum_{l<n,j,k,m_3,p} \frac{\delta_{n,2k+j+l}}{k!} \frac{(m_1+p)!}{\sqrt{m_1!m_3!}} \frac{\delta_{m_1+p,m_3+j-p}}{p!(j-p)!}$$

$$\times\, (\partial_X - i\partial_Y)^{k+j-p} (\partial_X + i\partial_Y)^p V(\mathbf{R}) (\partial_X + i\partial_Y)^k g^{(l)}_{m_3;m_2}(\mathbf{R})$$

The high field expansion:

Local equilibrium properties

How to get physical quantities at equilibrium?

Local observables: in terms of the exact eigenstates $|\Psi_\alpha\rangle$

- Electronic local density: $\rho(\mathbf{r}) = \sum_\alpha n_F(E_\alpha)|\langle\mathbf{r}|\Psi_\alpha\rangle|^2$
- Local current density: $\mathbf{j}(\mathbf{r}) = \sum_\alpha n_F(E_\alpha)\langle\Psi_\alpha|\hat{\mathbf{j}}(\mathbf{r})|\Psi_\alpha\rangle$

Electronic Green function:

$$G(\mathbf{r},\mathbf{r}',\omega) = \langle\mathbf{r}|(\omega - \hat{H}_0 - \hat{V})^{-1}|\mathbf{r}'\rangle = \sum_\alpha \frac{\Psi_\alpha^*(\mathbf{r}')\Psi_\alpha(\mathbf{r})}{\omega - E_\alpha}$$

$$\text{Clearly}: \ \rho(\mathbf{r}) = -\int \frac{d\omega}{\pi} n_F(\omega)\mathrm{Im}\,G(\mathbf{r},\mathbf{r},\omega + i0^+)$$

Magic formula: simple connection to vortex Green's functions

$$G(\mathbf{r},\mathbf{r}',\omega) = \int \frac{d^2\mathbf{R}}{2\pi l_B^2} \sum_{m,m'} \Psi^*_{m',\mathbf{R}}(\mathbf{r}')\Psi_{m,\mathbf{R}}(\mathbf{r}) \sum_{k=0}^{+\infty} \frac{1}{k!}\left(-\frac{l_B^2}{2}\Delta_{\mathbf{R}}\right)^k g_{m;m'}(\mathbf{R})$$

Electronic charge density

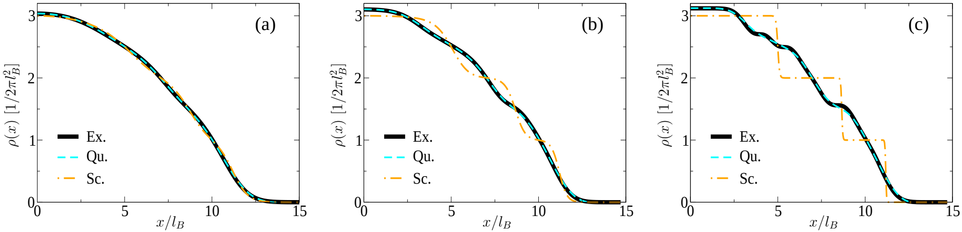
Quantum expansion: define $\xi_m(\mathbf{R}) = \epsilon_m + V(\mathbf{R}) - \mu$

$$\rho_{\mathrm{Qu.}}(\mathbf{r}) = \int \frac{d^2\mathbf{R}}{2\pi l_B^2} \sum_m n_F[\xi_m(\mathbf{R})] |\Psi_{m,\mathbf{R}}(\mathbf{r})|^2 + O(l_B^2)$$

Semiclassical result: point-like wavefunction for $l_B = 0$

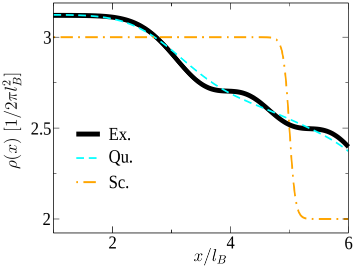
$$\rho_{\mathrm{Sc.}}(\mathbf{r}) = \frac{1}{2\pi l_B^2} \sum_m n_F[\xi_m(\mathbf{r})]$$

Checking a on solvable 1D model: for $k_B T/\hbar\omega_c = 0.2, 0.1, 0.01$



Full quantum result

Zooming in: deviations from terms like $(l_B^2 \Delta_{\mathbf{r}})^k \rho_{\mathrm{Qu.}}(\mathbf{r}) = O(1)$



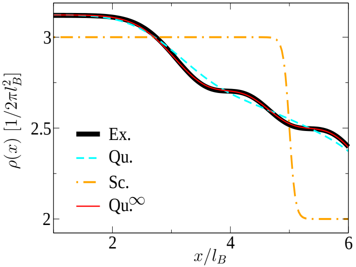
Infinite order resummation: it can be done!

$$\rho_{\mathrm{Qu.}\infty}(\mathbf{r}) = \int \frac{d^2\mathbf{R}}{2\pi l_B^2} \sum_{m=0}^{+\infty} \frac{n_F[\xi_m(\mathbf{R})]}{\pi m! l_B^2} A_m(\mathbf{R}-\mathbf{r}) \exp\left[-\frac{(\mathbf{R}-\mathbf{r})^2}{l_B^2}\right]$$

where $A_m(\mathbf{R}) = \frac{\partial^m}{\partial s^m}\left(\frac{1}{1+s}\exp\left[\frac{\mathbf{R}^2}{l_B^2}\frac{2s}{1+s}\right]\right)_{s=0}$: special polynomial

Full quantum result

Zooming in: deviations from terms like $(l_B^2 \Delta_{\mathbf{r}})^k \rho_{\mathrm{Qu.}}(\mathbf{r}) = O(1)$



Infinite order resummation: it can be done! IT WORKS!

$$\rho_{\mathrm{Qu.}\infty}(\mathbf{r}) = \int \frac{d^2\mathbf{R}}{2\pi l_B^2} \sum_{m=0}^{+\infty} \frac{n_F[\xi_m(\mathbf{R})]}{\pi m! l_B^2} A_m(\mathbf{R}-\mathbf{r}) \exp\left[-\frac{(\mathbf{R}-\mathbf{r})^2}{l_B^2}\right]$$

where $A_m(\mathbf{R}) = \frac{\partial^m}{\partial s^m}\left(\frac{1}{1+s}\exp\left[\frac{\mathbf{R}^2}{l_B^2}\frac{2s}{1+s}\right]\right)_{s=0}$: special polynomial

The high field limit in practice

Bottomline:

- Equilibrium local density $\rho(\mathbf{r})$ and current $\mathbf{j}(\mathbf{r})$ can be computed for all temperatures with excellent accuracy for any smooth potential
- Very simple density functional forms are obtained
- Possible use:
  - DFT-like calculations
  - Screening theory beyond the Thomas-Fermi approximation

Next question:

What about out of equilibrium transport?

Local current density: semiclassical result

Leading order: at $l_B \to 0$

- Drift current: bulk contribution

$$\mathbf{j}_{\mathbf{b}}^{(\mathbf{0})}(\mathbf{r}) = \frac{e}{h} \sum_{m=0}^{+\infty} n_F[\xi_m(\mathbf{r})] \nabla_{\mathbf{r}} V(\mathbf{r}) \times \hat{\mathbf{z}}$$

- Density gradient current: edge contribution

$$\mathbf{j}_{\mathbf{e}}^{(\mathbf{0})}(\mathbf{r}) = \frac{e}{h} \sum_{m=0}^{+\infty} \hbar\omega_c \left(m + \frac{1}{2}\right) \nabla_{\mathbf{r}} n_F[\xi_m(\mathbf{r})] \times \hat{\mathbf{z}}$$

Sub-leading order (bulk only): new terms!

$$\mathbf{j}_{\mathbf{b}}^{(\mathbf{2})}(\mathbf{r}) = l_B^2 \frac{e}{h} \sum_{m=0}^{+\infty} n_F[\xi_m(\mathbf{r})] \left[\frac{(\nabla_{\mathbf{r}} V \cdot \nabla_{\mathbf{r}})}{\hbar\omega_c} \nabla_{\mathbf{r}} V + \frac{3}{2}\left(m + \frac{1}{2}\right) \Delta_{\mathbf{r}} \nabla_{\mathbf{r}} V \right] \times \hat{\mathbf{z}}$$

The high field expansion:

Transport equations

Local equilibrium: Ohm's law

Total potential: $V(\mathbf{r}) = V_{\text{eff}}(\mathbf{r}) + e\Phi(\mathbf{r})$

- V_{eff}: confinement and impurity (screened) potential
- $\Phi(\mathbf{r})$: local out-of-equilibrium potential

Local conductivity tensor: purely transverse at $l_B \to 0$

- $\mathbf{j}(\mathbf{r}) = \hat{\sigma}(\mathbf{r})\mathbf{E} = -\sigma_H(\mathbf{r})\nabla\Phi(\mathbf{r}) \times \hat{\mathbf{z}}$
- $\sigma_H(\mathbf{r}) = \sum_m n_F[\epsilon_m + V_{\text{eff}}(\mathbf{r}) - \mu]$

Transport equation: $\nabla.\mathbf{j} = 0$ (continuity equation) gives

$$\big(\nabla\sigma_H(\mathbf{r}) \times \nabla\Phi(\mathbf{r})\big).\hat{\mathbf{z}} = 0$$

- Equipotentials coincide with lines of constant filling factor
- Indeterminacy at points where $\nabla\sigma_H(\mathbf{r}) = \mathbf{0}$

Conduction beyond the drift contribution

- Non-local contribution to current:

$$\delta \mathbf{j}(\mathbf{r}) = l_B^2 \frac{e^2}{h} \sum_{m=0}^{+\infty} n_F[\xi_m(\mathbf{r})] \frac{3}{2} \left(m + \frac{1}{2} \right) \Delta_{\mathbf{r}} \nabla_{\mathbf{r}} \Phi \times \hat{\mathbf{z}}$$

Originates from quantum tunneling

- Longitudinal and transverse corrections to the conductivity:

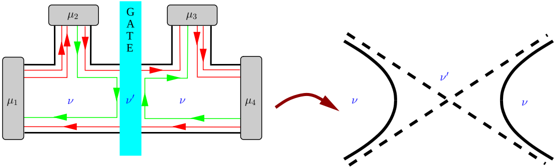
$$\delta \hat{\sigma}(\mathbf{r}) = \frac{l_B^2}{\hbar \omega_c} \sigma_H(\mathbf{r}) \begin{pmatrix} -\partial_{xy} V_{\mathrm{eff}} & \partial_{yy} V_{\mathrm{eff}} \\ \partial_{xx} V_{\mathrm{eff}} & \partial_{xy} V_{\mathrm{eff}} \end{pmatrix}$$

Local conductivities may not obey Onsager's relation!

Transport equation: keeping local terms only

$$(\nabla_{\mathbf{r}} \sigma_H \times \nabla_{\mathbf{r}} \Phi) \cdot \hat{\mathbf{z}} - \frac{l_B^2}{\hbar \omega_c} \sigma_H \mathrm{Tr} \left\{ \delta \hat{\sigma} . \begin{pmatrix} \partial_{xx} \Phi & \partial_{xy} \Phi \\ \partial_{xy} \Phi & \partial_{yy} \Phi \end{pmatrix} \right\} = 0$$

Checking bulk conduction against Büttiker picture



Model: $V_{\text{eff}}(\mathbf{r}) = V_{\text{eff}}(\mathbf{0}) + a\frac{x^2}{2} + b\frac{y^2}{2}$

- Non-trivial potential drop [for saddle point only $(ab < 0)$]:
  $\Phi(\mathbf{r}) = \left[A + B \int_0^{x/\lambda} dt \exp(-t^2)\right] \left[C + D \int_0^{y/\lambda} dt \exp(-t^2)\right]$
  where $\lambda^2 = -2\frac{l_B^2}{\hbar\omega_c} \frac{\sum_m n_F(\xi_m(\mathbf{0}))}{\sum_m n_F'(\xi_m(\mathbf{0}))}$
- Two-point conductance: $G_{2P} = \frac{e^2}{h}\sigma_H(\mathbf{0})$ Edge state result!

Remark: the conductance is independent of microscopic aspects

Transport: conduction vs. diffusion

Bottomline:

- Transport in IQHE regime may be investigated on the basis of simple bulk equations
- Microscopic details of the equilibrium density and current inhomogeneities are naturally taken into account: practical approach

Interesting directions to investigate:

- General connection to edge state formalism
- Study bulk transport equations for complex geometries (i.e. disordered)
- Role of non-local corrections: low temperature regime
- Coupling to self-consistent screening calculations

Conclusion

- Vortex wavefunctions are the naturally selected quantum states in the high field limit
- The mathematical foundation of vortex Green's functions was established
- Local equilibrium observables can be calculated accurately from simple density functionals
- Quasi-classical (high temperature) transport was investigated for a simplified scattering problem